Moonshine for Rudvalis's sporadic group I [*]

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Abstract

We introduce the notion of vertex operator superalgebra with enhanced conformal structure, which is a refinement of the notion of vertex operator superalgebra. We exhibit several examples, including a particular one which is self-dual, and whose full symmetry group is a direct product of a cyclic group of order seven with the sporadic simple group of Rudvalis. We thus obtain an analogue of Monstrous Moonshine for a sporadic group not involved in the Monster. Two variable analogues of the usual McKay–Thompson series are naturally associated to the action of the Rudvalis group on this object, and we provide explicit expressions for all the series arising.

Contents

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0 Introduction

We are interested in the problem of realizing sporadic simple groups as symmetry groups of vertex operator algebras (VOAs).

Perhaps the most striking example of this phenomena is the Moonshine VOA $V^\natural$ of [FLM88] which provides a realization of the Monster sporadic group $\mathbb{M}$. A feature of this example is that the Moonshine VOA conjecturally admits a characterization as the unique self-dual VOA of rank 24 with no (non-trivial) small (degree 1) elements. Assuming the conjecture is true, we then obtain an interesting description of the Monster group: as the symmetry group of this natural object in the category of VOAs, the unique self-dual VOA of rank 24 with no small elements.

Another feature of this example is the extent to which it elucidates a connection between the Monster group and the theory of modular forms; what has been referred to as Monstrous Moonshine since the historic article [CN79]. In particular, the fact that the Monster group acts as automorphisms of the Moonshine VOA allows one to attach a power series, a kind of $\mathbb{Z}$-graded character called the McKay–Thompson series, to each conjugacy class of $\mathbb{M}$. It is a non-trivial property of VOAs that these characters can be regarded as holomorphic functions on the upper half plane with certain modular properties [Zhu90]. The fact that the Moonshine VOA is self-dual (that is, has no irreducible modules other than itself) and has rank 24, entails that these functions are actually modular invariants with very special properties, although the theory of VOAs does perhaps not yet provide a full explanation for all the properties these series have been conjectured [CN79], and subsequently proven [Bor92] to satisfy.

A second example was given in [Höh96] where it was shown how one may modify the Moonshine VOA $V^\natural$ so as to arrive at a vertex operator superalgebra (VOSA) $VB^\natural$ whose full automorphism group is the direct product of a group of order two with the Baby Monster group $\mathbb{B}$. In this case also, one has a conjectural characterization for the VOSA $VB^\natural$, as the unique self-dual VOSA of rank $23\frac{1}{2}$ with no (non-trivial) small (degree 1 or degree 1/2) elements, and thus, conjecturally, we obtain an analogous description of the Baby Monster group: as the symmetry group of this object modulo its center. One may also consider the McKay–Thompson series associated to the Baby Monster group via its action on $VB^\natural$, so that one obtains another point of interaction between a sporadic simple group and the theory of modular forms.

Further examples of VOAs with finite simple symmetry groups have appeared in the literature. A VOA whose full automorphism group is $O^+_{10}(2)$ is studied in [Gri96], and the automorphism groups of closely related VOAs are determined in [Shi04] and [Shi06]. Results regarding the realization of 3-transposition groups as VOA symmetries have appeared in [KM01] and [Mat05]. Passing to positive characteristic, modular representations for certain sporadic groups involved in the Monster are constructed via VOAs over finite fields in [BR96]. This work arose from modular analogues of the Moonshine conjectures that were introduced in [Ryb94], and the proof of these conjectures was completed in [Bor98].

Evidence for a third example involving a sporadic simple group in characteristic zero appeared earlier in [FLM85] (see also [BR96]) and suggested a possible realization by vertex operators of the largest sporadic group of Conway Co_1. We considered this example in some detail in [Dun07] and found that although the Conway group does indeed act as symmetries, the full automorphism group of the VOSA structure alone is not finite. On the other hand, for a particular action of the Conway group on this VOSA there is a unique vector in the degree 3/2 space that is invariant for this group. What is more, the vertex operator associated to this element is such that its Fourier components, along with those of the usual conformal element (Virasoro element), generate an action of the $N=1$ Neveu–Schwarz super Virasoro algebra, a natural super analogue of the Virasoro algebra, acting on this VOSA. In [Dun07] we prove that the symmetry group fixing both the usual conformal element and this new superconformal element is precisely Conway's largest sporadic group. There we denote this object by $A^{f\natural}$, and we call it an $N=1$ VOSA in order to indicate that we regard it as one example from a family of vertex operator superalgebras for which the super Virasoro action is to be taken as axiomatic.

In direct analogy with $V^\natural$ and $VB^\natural$, one is quickly lead to conjecture the following characterization of $A^{f\natural}$: the unique self dual $N=1$ VOSA of rank 12 with no (non-trivial) small (degree 1/2) elements. We establish this conjecture, up to some technical conditions, in [Dun07].

Beyond ordinary VOSAs, $A^{f\natural}$ is our first example of what we now refer to as VOSAs with enhanced conformal structure, or more briefly, enhanced VOSAs. The idea is that we refine the notion of VOSA by imposing certain extra algebraic structure arising from distinguished vectors beyond the usual Virasoro and vacuum elements, and a precise definition is given in §2. One purpose of the present article is to motivate the notion of enhanced VOSA by furnishing further examples.

Our main example is a self-dual enhanced VOSA A_{Ru} of rank 28 whose full automorphism group is a direct product of a cyclic group of order seven with the sporadic simple group of Rudvalis. In

particular, the enhanced VOSA A_{Ru} furnishes a realization by vertex operators of one of the six sporadic groups not involved in the Monster.

In analogy with case of the Moonshine VOA and the Monster group we obtain a kind of moonshine for the Rudvalis group by considering the graded traces of elements of this group acting on A_{Ru}. In fact, the enhanced conformal structure on A_{Ru} is such that A_{Ru} admits a structure of (what we call) $U(1)$–VOSA, and to such objects one can naturally associate McKay–Thompson series in two variables, which specialize to the ordinary McKay–Thompson series after taking a suitable limit. The action of the Rudvalis group on A_{Ru} is sufficiently transparent that it is a straight forward task to provide explicit expressions for all the two variable McKay–Thompson series arising. We find that they are Jacobi forms for congruence subgroups of the modular group $SL_2(\mathbb{Z})$.

It will certainly be interesting to study the series associated to the Rudvalis group in more detail, and compare them to those associated to the Monster group via the Moonshine VOA, and to Conway's group via the enhanced VOSA studied in [Dun07].

The main result of this article is the following theorem.

Theorem (5.14)**.** *The quadruple $\mathcal{A}_{Ru} = (A_{Ru}, Y, \mathbf{1}, \Omega_{Ru})$ is a self-dual enhanced $U(1)$–VOSA of rank 28. The full automorphism group of (A_{Ru}, Ω_{Ru}) is a direct product of a cyclic group of order seven with the sporadic simple group of Rudvalis.*

In light of the above discussion, it is natural to hope that VOA theory might be used to furnish convenient characterizations of certain finite simple groups. In the case of the Conway group and $A^{f\natural}$, the enhanced conformal structure was essential to the characterization, and we regard this as another motivation for the notion of enhanced VOSA.

Later in the article we conjecture a uniqueness result for $\mathcal{A}_{Ru}$, which is analogous to those which hold for the Golay code, the Leech lattice (see [Con69]), and the $N = 1$ VOSA for Conway's group, and those which are conjectured to hold for the Moonshine VOA and the Baby Monster VOSA. All these objects have sporadic automorphism groups. The object $\mathcal{A}_{Ru}$ is a first example with non-Monstrous sporadic automorphism group.

0.1 Outline

In §1 we review some facts about VOAs[1]. We recall some basics from the formal calculus in §1.1, and then the definition of VOAs in §1.2. We also recall from [FLM88] the higher order brackets defined on a VOA in §1.3. These higher order brackets are generalizations of the ordinary Lie bracket, and are natural in the context of VOA theory, and to some extent they motivate the notion of vertex Lie algebra which we review in §1.4. The notion of vertex Lie algebra plays a role in the definition of enhanced VOA, and this in turn appears in §2.

In §3.1 we review our conventions regarding Clifford algebras; in §3.2 the Spin groups, and in §3.3, Clifford algebra modules. In §3.4 we review the well-known construction of VOA structure on

[1] Here, and from here on, we will usually suppress the "super" in super-objects, so that unless extra clarification is necessary, superspaces and superalgebras will be referred to as spaces and algebras, respectively, and the term VOA for example, will be used even when the underlying vector space comes equipped with a $\mathbb{Z}/2$-grading.

certain infinite dimensional Clifford algebra modules constructed in turn from a finite dimensional vector space with symmetric bilinear form, and in §3.5 we make some conventions regarding the Hermitian structures arising in the case that the initial vector space comes equipped also with a suitable Hermitian form.

In §4 we present a family of enhanced VOAs whose full automorphism groups are the general linear groups $GL_N(\mathbb{C})$. We later require to realize symmetry groups no larger than $SL_N(\mathbb{C})$, and for this it is useful to consider a twisted analogue of the Clifford module VOA construction given in §3.4. We review such a construction in §4.2, and then in §4.3 we consider the specific example of an enhanced VOA with symmetry group of the form $SL_{28}(\mathbb{C})/\langle\pm\mathrm{Id}\rangle$.

In §5 we construct an enhanced VOA whose automorphism group is a sevenfold cover of the sporadic simple group of Rudvalis. We give two constructions of the enhanced conformal structure. The first construction, in §5.1, arises directly from the geometry of the Conway–Wales lattice, which was introduced in [Con77]. This approach is more brief and more conceptual than the second construction, in §5.2, which is nonetheless more convenient for (computer aided) computations. The approach of §5.2 involves analysis of a certain monomial group that is closely related to the Cayley algebra (or integral octonion algebra) structure with which one may equip the E_8 lattice. We review this structure in §5.2.1, then the monomial group is constructed in §5.2.2, and we determine in §5.2.3 the invariants of this monomial group under its action on a certain space. Sections 5.1 and 5.2 are independent. Picking up from the end of either one, the construction of the enhanced VOA for the Rudvalis group is given in §5.3. The symmetry group is considered in §5.4, and we discuss a conjectural characterization of the enhanced VOA for the Rudvalis group at the very end of this section.

In §6 we furnish explicit expressions for the two variable McKay–Thompson series associated to A_{Ru}. In particular, we introduce the notion of two variable McKay–Thompson series in §6.4, and we consider the two variable series arising for the Rudvalis group in §6.6. The Appendix contains tables displaying the first few coefficients of each two variable McKay–Thompson series arising from the action of the Rudvalis group on A_{Ru}.

0.2 Notation

We choose a square root of -1 in $\mathbb{C}$ and denote it by $\mathbf{i}$. For q a prime power $\mathbb{F}_q$ shall denote a field with q elements. We use $\mathbb{F}^\times$ to denote the non-zero elements of a field $\mathbb{F}$. More generally, $A^\times$ shall denote the set of invertible elements in an algebra A. For G a group and $\mathbb{F}$ a field we write $\mathbb{F}G$ for the group algebra of G over $\mathbb{F}$. For the remainder we shall use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

For Σ a finite set, we denote the power set of Σ by $\mathcal{P}(\Sigma)$. The set operation of symmetric difference equips $\mathcal{P}(\Sigma)$ with a structure of $\mathbb{F}_2$-vector space, and with this structure in mind, we sometimes write $\mathbb{F}_2^\Sigma$ in place of $\mathcal{P}(\Sigma)$. Suppose that Σ has N elements. The space $\mathbb{F}_2^\Sigma$ comes equipped with a function $\mathbb{F}_2^\Sigma \to \{0,1,\ldots,N\}$ called *weight*, which assigns to an element $\gamma\in\mathbb{F}_2^\Sigma$ the cardinality of the corresponding element of $\mathcal{P}(\Sigma)$. An $\mathbb{F}_2$-subspace of $\mathbb{F}_2^\Sigma$ is called a *binary linear code of length N*. A binary linear code $\mathcal{C}$ is called *even* if all its codewords have even weight, and is called *doubly even* if all codewords have weight divisible by 4. The space $\mathbb{F}_2^\Sigma$ carries a bilinear form defined by setting $\langle C,D\rangle=\sum_i\gamma_i\delta_i$ for $C=(\gamma_i)$ and $D=(\delta_i)$, so that given a binary linear code

$\mathcal{C}$ we may consider the dual code $\mathcal{C}^\circ$ defined by

$$\mathcal{C}^\circ = \left\{ D \in \mathbb{F}_2^\Sigma \mid \langle C, D \rangle = 0, \forall C \in \mathcal{C} \right\} \tag{0.2.1}$$

We say that $\mathcal{C}$ is *self-dual* in the case that $\mathcal{C} = \mathcal{C}^\circ$, and we say that $\mathcal{C}$ is *self-orthogonal* when $\mathcal{C} \subset \mathcal{C}^\circ$. We write $\mathcal{C}^*$ for the *co-code* $\mathcal{C}^* = \mathbb{F}_p^\Sigma / \mathcal{C}$.

A *superspace* is a vector space with a grading by $\mathbb{Z}/2 = \{\bar{0}, \bar{1}\}$. When M is a superspace, we write $M = M_{\bar{0}} \oplus M_{\bar{1}}$ for the superspace decomposition, and for $u \in M$ we set $|u| = \gamma \in \{\bar{0}, \bar{1}\}$ when u is $\mathbb{Z}/2$-homogeneous and $u \in M_\gamma$. The dual space M^* has a natural superspace structure such that $(M^*)_\gamma = (M_\gamma)^*$ for $\gamma \in \{\bar{0}, \bar{1}\}$. The space $\mathrm{End}(M)$ admits a structure of Lie superalgebra when equipped with the Lie superbracket $[\cdot\,,\cdot]$ which is defined so that $[a, b] = ab - (-1)^{|a||b|} ba$ for $\mathbb{Z}/2$-homogeneous a, b in $\mathrm{End}(M)$.

Almost all vector spaces in this article will be most conveniently regarded as superspaces (with possibly trivial odd parts) and similarly for algebras, and so from this point onwards, and unless otherwise qualified we will use the terms "space" and "algebra" as inclusive of the notions of superspace and superalgebra, respectively. Thus we will speak of vertex operator algebras, and Lie algebras, and so on, even when the underlying vector spaces are $\mathbb{Z}/2$ graded.

When z denotes a formal variable, we write $M[[z]]$ for the space of formal Taylor series with coefficients in M, and $M((z))$ for the space of formal Laurent series with coefficients in M. All formal variables in this article will be regarded as even, so that the superspace decomposition of $M[[z]]$ is $M_{\bar{0}}[[z]] \oplus M_{\bar{1}}[[z]]$, and similarly for $M((z))$. We write $\bigwedge(M)$ for the full exterior algebra of a vector space M. We write $\bigwedge(M) = \bigwedge(M)^0 \oplus \bigwedge(M)^1$ for the parity decomposition of $\bigwedge(M)$, and we write $\bigwedge(M) = \bigoplus_{k \geq 0} \bigwedge^k(M)$ for the natural $\mathbb{Z}$-grading on $\bigwedge(M)$. We denote by D_z the operator on formal power series which is formal differentiation in the variable z, so that if $f(z) = \sum c_m z^{-m-1} \in M((z))$ is a formal power-series with coefficients in some space M, we have $D_z f(z) = \sum (-m) f_{m-1} z^{-m-1}$. For m a non-negative integer, we set $D_z^{(m)} = \frac{1}{m!} D_z^m$.

As is customary, we use $\eta(\tau)$ to denote the Dedekind eta function.

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{0.2.2}$$

Here $q = e^{2\pi \mathbf{i} \tau}$ and τ is a variable in the upper half plane, which we denote by $\mathbf{h}$. Recall also the Jacobi theta function $\vartheta_3(\xi|\tau)$ defined by

$$\vartheta_3(\xi|\tau) = \sum_{m \in \mathbb{Z}} e^{2\mathbf{i}\xi m + \pi \mathbf{i} \tau m^2} \tag{0.2.3}$$

for $\tau \in \mathbf{h}$ and $\xi \in \mathbb{C}$. According to the Jacobi Triple Product Identity we have

$$\vartheta_3(\xi|\tau) = \prod_{m \geq 0} (1 - q^{m+1})(1 + e^{2\mathbf{i}\xi} q^{m+1/2})(1 + e^{-2\mathbf{i}\xi} q^{m+1/2}) \tag{0.2.4}$$

with $q = e^{2\pi \mathbf{i} \tau}$ as before.

The modular group $\bar{\Gamma} = PSL_2(\mathbb{Z})$ acts in a natural way on the upper half plane $\mathbf{h}$. This action is generated by the modular transformations $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$, which we denote by S and

T, respectively, and these generators are subject to the relations $S^2 = (ST)^3 = \mathrm{Id}$. We write $\bar{\Gamma}_\theta$ for the theta group: the subgroup of $\bar{\Gamma}$ generated by S and T^2.

The most specialized notations arise in §3. We include here a list of them, with the relevant subsections indicated in brackets. They are grouped roughly according to similarity of appearance, rather than by order of appearance, so that the list may be easier to search through, whenever the need might arise.

$\mathfrak{a}$ A complex vector space with non-degenerate Hermitian form (§3.5).

$\mathfrak{a}^*$ The dual space to $\mathfrak{a}$ (§3.5).

$\mathfrak{u}$ A real or complex vector space of even dimension with non-degenerate bilinear form, assumed to be positive definite in the real case (§3.1). In the case that $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$, the bilinear form is assumed to be $1/2$ times the symmetric linear extension of the natural pairing between $\mathfrak{a}$ and $\mathfrak{a}^*$ (§3.5).

$\{a_i\}_{i\in\Delta}$ A basis for $\mathfrak{a}$, orthonormal in the sense that $(a_i, a_j) = \delta_{ij}$ for $i, j \in \Delta$ (§3.5).

$\{a_i^*\}_{i\in\Delta}$ The dual basis to $\{a_i\}_{i\in\Delta}$ (§3.5).

$\{e_i\}_{i\in\Sigma}$ A basis for $\mathfrak{u}$, orthonormal in the sense that $\langle e_i, e_j\rangle = \delta_{ij}$ for $i, j \in \Sigma$ (§3.1). In the case that $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$ we set $\Sigma = \Delta \cup \Delta'$, and we insist that $e_i = a_i + a_i^*$ and $e_{i'} = \mathbf{i}(a_i - a_i^*)$ for $i \in \Delta$ (§3.5).

e_I We write e_I for $e_{i_1} \cdots e_{i_k} \in \mathrm{Cliff}(\mathfrak{u})$ when $I = \{i_1, \ldots, i_k\}$ is a subset of Σ and $i_1 < \cdots < i_k$ (§3.1).

$g(\cdot)$ We write $g \mapsto g(\cdot)$ for the natural homomorphism $\mathit{Spin}(\mathfrak{u}) \to \mathit{SO}(\mathfrak{u})$. Regarding $g \in \mathit{Spin}(\mathfrak{u})$ as an element of $\mathrm{Cliff}(\mathfrak{u})^\times$ we have $g(u) = gug^{-1}$ in $\mathrm{Cliff}(\mathfrak{u})$ for $u \in \mathfrak{u}$. More generally, we write $g(x)$ for gxg^{-1} when x is any element of $\mathrm{Cliff}(\mathfrak{u})$.

$e_I(\cdot)$ When I is even, $e_I \in \mathrm{Cliff}(\mathfrak{u})$ lies also in $\mathit{Spin}(\mathfrak{u})$, and $e_I(x)$ denotes $e_I x e_I^{-1}$ when $x \in \mathrm{Cliff}(\mathfrak{u})$.

$\mathfrak{z}$ We denote $e_\Sigma \in \mathit{Spin}(\mathfrak{u})$ also by $\mathfrak{z}$ (§3.2).

θ The map which is $-\mathrm{Id}$ on $\mathfrak{u}$, or the parity involution on $\mathrm{Cliff}(\mathfrak{u})$ (§3.1), or the parity involution on $A(\mathfrak{u})_\Theta$ (§3.4).

$\theta^{1/2}$ The map which is $\mathbf{i}\,\mathrm{Id}$ on $\mathfrak{a}$ and $-\mathbf{i}\,\mathrm{Id}$ on $\mathfrak{a}^*$, or a lift of this map to $\mathrm{Cliff}(\mathfrak{u})$ or $A(\mathfrak{u})_\Theta$ (§3.5).

1_E A vector in $\mathrm{CM}(\mathfrak{u})_E$ such that $x1_E = 1_E$ for x in E (§3.3).

$\mathbf{1}_E$ The vector corresponding to 1_E under the identification between $\mathrm{CM}(\mathfrak{u})_E$ and $(A(\mathfrak{u})_{\theta,E})_{N/8}$ when $\mathfrak{u}$ has dimension $2N$ (§3.4).

Δ A finite ordered set indexing an orthonormal basis for $\mathfrak{a}$ (§3.5).

Δ' A second copy of Δ with the natural identification $\Delta \leftrightarrow \Delta'$ denoted $i \leftrightarrow i'$ for $i \in \Delta$ (§3.5).

Σ A finite ordered set indexing an orthonormal basis for $\mathfrak{u}$ (§3.1). In the case that $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$, we set $\Sigma = \Delta \cup \Delta'$ and insist that Σ be ordered according to the ordering on Δ and the rule $i < j'$ for $i, j \in \Delta$ (§3.5).

$\mathcal{E}$ A label for the basis $\{e_i\}_{i\in\Sigma}$ (§3.1).

E A subgroup of $\mathrm{Cliff}(\mathfrak{u})^\times$ homogeneous with respect to the $\mathbb{F}_2^{\mathcal{E}}$ grading on $\mathrm{Cliff}(\mathfrak{u})$ (§3.3).

X In the case that $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$, we take $E = X$ in the above (and the below), where X is the subgroup of $\mathrm{Cliff}(\mathfrak{u})$ generated by elements $\mathbf{i}e_i e_{i'}$ for $i \in \Delta$ (§3.5).

$A(\mathfrak{u})$ The Clifford module VOA associated to the vector space $\mathfrak{u}$ (§3.4).

$\tilde{A}(\mathfrak{u})$ The twisted Clifford module VOA associated to $\mathfrak{u}$, obtained by taking θ-fixed points of $A(\mathfrak{u})_\Theta$ (§4.2).

$A(\mathfrak{u})_\theta$ The canonically θ-twisted module over $A(\mathfrak{u})$ (§3.4).

$A(\mathfrak{u})_{\theta,E}$ The θ-twisted module $A(\mathfrak{u})_\theta$ realized in such a way that the subspace of minimal degree may be identified with $\mathrm{CM}(\mathfrak{u})_E$ (§3.4).

$A(\mathfrak{u})_\Theta$ The direct sum of $A(\mathfrak{u})$-modules $A(\mathfrak{u}) \oplus A(\mathfrak{u})_\theta$ (§3.4).

$\mathcal{C}(E)$ The binary linear code on Σ consisting of elements I in $\mathbb{F}_2^\Sigma$ for which E has non-trivial intersection with $\mathbb{F}e_I \subset \mathrm{Cliff}(\mathfrak{u})$ (§3.3).

$\mathrm{CM}(\mathfrak{u})_E$ The module over $\mathrm{Cliff}(\mathfrak{u})$ induced from a trivial module over E (§3.3).

$\mathrm{Cliff}(\mathfrak{u})$ The Clifford algebra associated to the vector space $\mathfrak{u}$ (§3.1).

$\mathit{Spin}(\mathfrak{u})$ The spin group associated to the vector space $\mathfrak{u}$ (§3.2).

$\langle\cdot\,,\cdot\rangle$ A non-degenerate symmetric bilinear form on $\mathfrak{u}$ or on $\mathrm{Cliff}(\mathfrak{u})$ (§3.1), or on the $\mathrm{Cliff}(\mathfrak{u})$-module $\mathrm{CM}(\mathfrak{u})_E$ (§3.3). In the case that $\mathfrak{u}$ is real all of these forms will be positive definite.

$(\cdot\,,\cdot)$ A non-degenerate Hermitian form on $\mathfrak{a}$, or on $\mathrm{Cliff}(\mathfrak{u})$ or $\mathrm{CM}(\mathfrak{u})_X$ in the case that $\mathfrak{u}=\mathfrak{a}\oplus\mathfrak{a}^*$ (§3.5). The Hermitian forms arising will always be anti-linear in the right hand slot.

$\langle\,\cdot\,|\,\cdot\,\rangle$ A non-degenerate symmetric bilinear form on $A(\mathfrak{u})_\Theta$ (§3.4).

1 Vertex operator algebras

In §1.2 we review the definition of VOA, after recalling some facts from the formal calculus in §1.1. In §1.3 we recall from [FLM88] the higher order generalizations of the Lie bracket, which arise naturally in the context of VOAs, and in §1.4 we recall some facts about vertex Lie algebras (also known as conformal algebras).

1.1 Formal calculus

Given a rational function $f(z,w)$ with poles only at $z=0$, $w=0$, and $z=w$, we write $\iota_{z,w}f(z,w)$, $\iota_{w,z}f(z,w)$, and $\iota_{w,z-w}f(z,w)$ for the power series expansions of $f(z,w)$ in the respective domains: $|z|>|w|>0$, $|w|>|z|>0$, and $|w|>|z-w|>0$. Then in the case that $f(z,w)=z^mw^n(z-w)^l$ for some $m,n,l\in\mathbb{Z}$ for example, we have

$$\iota_{z,w}f(z,w)=\sum_{k\geq 0}(-1)^k\binom{l}{k}z^{m+l-k}w^{n+k} \tag{1.1.1}$$

$$\iota_{w,z}f(z,w)=\sum_{k\geq 0}(-1)^{l+k}\binom{l}{k}z^{m+k}w^{n+l-k} \tag{1.1.2}$$

$$\iota_{w,z-w}f(z,w)=\sum_{k\geq 0}\binom{m}{k}w^{m+n-k}(z-w)^{l+k} \tag{1.1.3}$$

For $f(z_1,\ldots,z_k)=\sum c_{m_1,\ldots,m_k}z_1^{-m_1-1}\cdots z_k^{-m_k-1}$ a formal power series in variables z_i, we write $\mathrm{Res}_{z_i}f$ for the coefficient of z_i^{-1} in $f(z_1,\ldots,z_k)$, so that

$$\mathrm{Res}_{z_1}f(z_1,\ldots,z_k)=\sum c_{0,m_2,\ldots,m_k}z^{-m_2-1}\cdots z^{-m_k-1} \tag{1.1.4}$$

for example, and we write $\mathrm{Sing}\,f$ for the sum of terms involving only negative powers of all the variables z_i.

$$\mathrm{Sing}\,f(z_1,\ldots,z_k)=\sum_{m_1,\ldots,m_k\geq 0}c_{m_1,\ldots,m_k}z^{-m_1-1}\cdots z^{-m_k-1} \tag{1.1.5}$$

Suppose again that $f(z,w)$ is a rational function with possible poles at $z=0$, $w=0$, and $z=w$. Then we have the following special case of the Cauchy Theorem

$$\int_{C_R(0)}f(z,w)\mathrm{d}z-\int_{C_r(0)}f(z,w)\mathrm{d}z=\int_{C_\epsilon(w)}f(z,w)\mathrm{d}z \tag{1.1.6}$$

where $C_a(z_0)$ denotes a positively oriented circular contour (in the z-plane) of radius a about z_0, and R, r and ϵ are chosen so that $R > |w| > r > 0$, and $\epsilon < \min\{R - |w|, |w| - r\}$. The following identity is then an algebraic reformulation of (1.1.6).

$$\mathrm{Res}_z\, \iota_{z,w} f(z,w) - \mathrm{Res}_z\, \iota_{w,z} f(z,w) = \mathrm{Res}_{z-w}\, \iota_{w,z-w} f(z,w) \tag{1.1.7}$$

1.2 Structure

For a *vertex algebra* structure on a (super)space $U = U_{\bar{0}} \oplus U_{\bar{1}}$ over a field $\mathbb{F}$ we require the following data.

- *Vertex operators:* an even morphism $Y : U \otimes U \to U((z))$ such that when we write $Y(u,z)v = \sum_{n\in\mathbb{Z}} u_{(n)} v z^{-n-1}$, we have $Y(u,z) = 0$ only when $u = 0$.
- *Vacuum:* a distinguished vector $\mathbf{1} \in U_{\bar{0}}$ such that $Y(\mathbf{1},z)u = u$ for $u \in U$, and $Y(u,z)\mathbf{1}|_{z=0} = u$.

This data furnishes a vertex algebra structure on U just when the following identity is satisfied.

- *Jacobi identity:* for $\mathbb{Z}/2$ homogeneous $u, v \in U$, and for any $m, n, l \in \mathbb{Z}$ we have
$$\begin{aligned} &\mathrm{Res}_z\, Y(u,z)Y(v,w)\iota_{z,w}F(z,w) \\ &- \mathrm{Res}_z (-1)^{|u||v|} Y(v,w)Y(u,z)\iota_{w,z}F(z,w) \\ &= \mathrm{Res}_{z-w}\, Y(Y(u,z-w)v,w)\iota_{w,z-w}F(z,w) \end{aligned} \tag{1.2.1}$$
where $F(z,w) = z^m w^n (z-w)^l$.

We denote such an object by the triple $(U, Y, \mathbf{1})$.

A *vertex operator algebra (VOA)* is vertex algebra $(U, Y, \mathbf{1})$ equipped with a distinguished element $\omega \in U_{\bar{0}}$ called the *Virasoro element*, such that $L(-1) := \omega_{(0)}$ satisfies

$$[L(-1), Y(u,z)] = D_z Y(u,z) \tag{1.2.2}$$

for all $u \in U$, and such that the operators $L(n) := \omega_{(n+1)}$ furnish a representation of the Virasoro algebra on the vector space underlying U, so that we have

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0} c \mathrm{Id} \tag{1.2.3}$$

for some $c \in \mathbb{F}$. We also require the following grading condition.

- $L(0)$-*grading:* the action of $L(0)$ on U is diagonalizable with rational eigenvalues bounded from below, and is such that the $L(0)$-homogeneous subspaces $U_n := \{u \in U \mid L(0)u = nu\}$ are finite dimensional.

When these conditions are satisfied we write $(U, Y, \mathbf{1}, \omega)$ in order to indicate the particular data that constitutes the VOA structure on U. The value c in (1.2.3) is called the *rank* of U, and we denote it by $\mathrm{rank}(U)$.

Following [Höh96] we say that a VOA U is *nice* when the eigenvalues of $L(0)$ are non-negative and contained in $\frac{1}{2}\mathbb{Z}$, and the degree zero subspace U_0 is spanned by the vacuum vector $\mathbf{1}$. All the VOAs we consider in this paper will be nice VOAs.

We refer the reader to [FHL93] for a discussion of VOA modules, twisted modules, adjoint operators, and intertwining operators. A VOA is said to be *self-dual* in the case that it has no non-trivial irreducible modules other than itself.

1.3 Higher order brackets

Let $(U, Y, \mathbf{1}, \omega)$ be a VOA, and let $u, v \in U$. Then taking $F(z, w) = z^m$ in the Jacobi identity for VOAs (1.2.1) we obtain

$$[u_{(m)}, Y(v, w)] = \sum_{k \geq 0} \binom{m}{k} Y(u_{(k)}v, w)w^{m-k} \tag{1.3.1}$$

and from this we may derive the following formula for the Lie (super)bracket of two vertex operators $Y(u, z)$ and $Y(v, w)$.

$$[Y(u, z), Y(v, w)] = \sum_{m} \sum_{k \geq 0} \binom{m}{k} Y(u_{(k)}v, w)w^{m-k}z^{-m-1} \tag{1.3.2}$$

More generally, there are a family of products on vertex operators $Y(u, z)$ and $Y(v, w)$ defined by setting

$$[Y(u, z) \times_n Y(v, w)] = (z - w)^n [Y(u, z), Y(v, w)] \tag{1.3.3}$$

for n a non-negative integer. We call these products the n^{th}*-order brackets*, so that the 0^{th}-order bracket is the usual Lie bracket. Taking $F(z, w) = z^m(z - w)^n$ in the Jacobi identity (1.2.1) we obtain the following identity for the n^{th}-order brackets.

$$[Y(u, z) \times_n Y(v, w)] = \sum_{m} \sum_{k \geq 0} \binom{m}{k} Y(u_{(k+n)}v, w)w^{m-k}z^{-m-1} \tag{1.3.4}$$

On the other hand, the Jacobi identity for VOAs also entails that for any $u, v, a \in U$, the expressions $Y(u, z)Y(v, w)a$ and $Y(Y(u, z-w)v, w)a$ may be viewed as expansions of a common element of the space

$$U[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \tag{1.3.5}$$

in the domains $|z| > |w| > 0$ and $|w| > |z - w| > 0$, respectively (c.f [FHL93]). With this understanding of "equality" we might write

$$Y(u, z)Y(v, w) = \sum_{k} Y(u_{(k)}v, w)(z - w)^{-k-1}. \tag{1.3.6}$$

Then, by (1.3.2), the data on the right hand side of (1.3.6) suffices for the purpose of computing the commutator $[Y(u,z),Y(v,w)]$. In fact, only the terms with positive k contribute to $[Y(u,z),Y(v,w)]$, and thus it is common to write simply

$$Y(u,z)Y(v,w)=\sum_{k\geq 0}\frac{Y(u_{(k)}v,w)}{(z-w)^{k+1}}+\text{reg.} \tag{1.3.7}$$

where "reg." denotes regular terms in $(z-w)$. This expression is called the *operator product expansion (OPE)* for $Y(u,z)Y(v,w)$, and we have seen that the bracket $[Y(u,z),Y(v,w)]$ can be easily computed once this OPE is given. Similarly for the n^{th}-order bracket, it is clear from (1.3.4) that it is sufficient to specify the expression

$$(z-w)^nY(u,z)Y(v,w)=\sum_{k\geq 0}\frac{Y(u_{(k+n)}v,w)}{(z-w)^{k+1}}+\text{reg.} \tag{1.3.8}$$

which we call the n^{th}-order OPE, in order to compute $[Y(u,z)\times_n Y(v,w)]$.

We may define products $[u\times_n v]_{kl}$ for $u,v\in U$ and $k,l\in\mathbb{Z}$ by setting

$$[Y(u,z)\times_n Y(v,w)]=\sum_{k,l}\,[u\times_n v]_{kl}\;z^{-k-1}w^{-l-1} \tag{1.3.9}$$

Then one may check directly from the definition that the component operators $[u\times_n v]_{kl}$ may be expressed as follows in terms of the usual Lie bracket.

$$[u\times_n v]_{kl}=\sum_{m\geq 0}(-1)^m\binom{n}{m}[u_{(k+n-m)},v_{(l+m)}] \tag{1.3.10}$$

1.4 Vertex Lie algebras

From §1.3 we see that much of the algebra structure on a VOA (and similarly for vertex algebras) is encoded in the singular terms of the operators $Y(u,z)v$. The notion of vertex Lie algebra is an axiomatic formulation of the kind of object one obtains by replacing $Y(u,z)v$ with $\operatorname{Sing}Y(u,z)v$ in the definition of vertex algebra (recall §1.1). More precisely, for a structure of vertex Lie algebra on a superspace $R=R_{\bar 0}\oplus R_{\bar 1}$ we require morphisms $Y_-:R\otimes R\to z^{-1}R[z^{-1}]$ and $T:R\to R$ such that the following axioms are satisfied.

- *Translation:* $Y_-(Tu,z)=D_zY(u,z)$.
- *Skew–symmetry:* $Y_-(u,z)v=\operatorname{Sing}e^{zT}Y_-(v,-z)u$.
- *Jacobi identity:* for $\mathbb{Z}/2$ homogeneous $u,v\in R$, and for any $m,n,l\in\mathbb{Z}$ we have

$$\begin{aligned}&\operatorname{Sing}\operatorname{Res}_z Y_-(u,z)Y_-(v,w)\iota_{z,w}F(z,w)\\&-\operatorname{Sing}\operatorname{Res}_z(-1)^{|u||v|}Y_-(v,w)Y_-(u,z)\iota_{w,z}F(z,w)\\&=\operatorname{Sing}\operatorname{Res}_{z-w}Y_-(Y_-(u,z-w)v,w)\iota_{w,z-w}F(z,w)\end{aligned} \tag{1.4.1}$$

  where $F(z,w)=z^mw^n(z-w)^l$.

We denote such an object by $R = (R, Y_-, T)$. We write the image of $u \otimes v$ under Y_- as $Y_-(u,z)v = \sum_{n\geq 0} u_{(n)} v z^{-n-1}$. We then obtain an equivalent definition by replacing the Jacobi identity with the following commutativity requirement (c.f. [Pri99],[Li04]).

$$[u_{(m)}, Y_-(v,z)] = \sum_{n\geq 0} \binom{m}{n} \operatorname{Sing} Y_-(u_{(n)}v, w) w^{m-n} \tag{1.4.2}$$

Vertex Lie algebras were introduced independently by V. Kac [Kac98] (where they are called *conformal algebras*) and M. Primc [Pri99]. They have been studied extensively by V. Kac and his collaborators, and we refer the reader to [Kac99], [BKV99], [BDK01], and the references therein for many interesting results.

As hinted at above, for any vertex algebra $U = (U, Y, \mathbf{1})$ we obtain a vertex Lie algebra (U, Y_-, T) by setting $Y_- = \operatorname{Sing} Y$, and by setting T to be the *translation operator:* the morphism on U defined so that $Tu = u_{-2}\mathbf{1}$ for $u \in U$. (If $(U, Y, \mathbf{1}, \omega)$ is a VOA, then T so defined satisfies $T = L(-1)$.) We abuse notation somewhat to write $\operatorname{Sing} U$ for this object $(U, \operatorname{Sing} Y, T)$. Conversely, to any vertex Lie algebra one may canonically associate a vertex algebra called the *enveloping vertex algebra*, and this construction plays an analogous role for vertex Lie algebras as universal enveloping algebras do for ordinary Lie algebras.

On the other hand, to each vertex Lie algebra R is canonically associated a Lie algebra $\operatorname{Lie}(R)$ called the *local Lie algebra* of R. As a vector space, we have $\operatorname{Lie}(R) = R[t,t^{-1}]/\operatorname{Im}\partial$ where ∂ is the operator $T \otimes 1 + \mathrm{Id}_R \otimes D_t$ on $R[t,t^{-1}] = R \otimes \mathbb{F}[t,t^{-1}]$. Writing $u_{[m]}$ for the image of $u \otimes t^m$ in $\operatorname{Lie}(R)$ we have

Proposition 1.1. $\operatorname{Lie}(R)$ *is a Lie algebra under the Lie bracket*

$$\left[u_{[m]}, v_{[n]}\right] = \sum_{k\geq 0} \binom{m}{k} (u_{(k)}v)_{[m+n-k]} \tag{1.4.3}$$

and $(Tu)_{[m]} = -mu_{[m-1]}$ *for all* u *in* R.

Example. In the case that $R = \operatorname{Sing} U$ for U a vertex algebra, Proposition 1.1 furnishes a Lie algebra structure on the abstract space spanned by Fourier coefficients $u_{(m)}$ for $u \in U$, $m \in \mathbb{Z}$, and subject to the relations $(Tu)_{(m)} = -mu_{(m-1)}$. One can define a kind of enveloping algebra for the vertex algebra U by taking a certain quotient of (a suitable completion of) the universal enveloping algebra of $\operatorname{Lie}(R)$.

Given a vertex Lie algebra R and a subset $\Omega \subset R$ we may consider the *vertex Lie subalgebra generated by* Ω. This is by definition just the intersection of all vertex Lie subalgebras of R that contain Ω. When $\Omega \subset U$ for some vertex algebra $(U, Y, \mathbf{1})$, we will write $[\Omega]$ for the vertex Lie subalgebra of $\operatorname{Sing} U$ generated by Ω.

Example. Suppose $(U, Y, \mathbf{1}, \omega)$ is a VOA. Set $\Omega = \{\omega\}$ and let $R = [\Omega]$ be the vertex Lie subalgebra of $\operatorname{Sing} U$ generated by Ω. Then we have

$$R = \operatorname{Span}\{\mathbf{1}, L(-1)^k\omega \mid k \geq 0\} \tag{1.4.4}$$

and $\operatorname{Lie}(R)$ is a copy of the Virasoro algebra.

The following result is straightforward.

Proposition 1.2. *If* $U = (U, Y, \mathbf{1})$ *is a vertex algebra and* R *is a vertex Lie subalgebra of* $\operatorname{Sing} U$*, then the map* $\operatorname{Lie}(R) \to \operatorname{End}(U)$ *given by* $u_{[m]} \mapsto u_{(m)}$ *is a (well-defined) homomorphism of Lie algebras. In particular,* $\operatorname{Lie}(R)$ *is canonically represented on* U*.*

2 Enhanced vertex algebras

We begin with the

Definition. An *enhanced vertex algebra* is a quadruple $(U, Y, \mathbf{1}, R)$ such that $(U, Y, \mathbf{1})$ is a vertex algebra, and R is a vertex Lie subalgebra of $\operatorname{Sing}(U, Y, \mathbf{1})$. We say that $(U, Y, \mathbf{1}, R)$ is an *enhanced vertex operator algebra (enhanced VOA)* if there is a unique $\omega \in R$ such that

1. $(U, Y, \mathbf{1}, \omega)$ is a VOA, and
2. $\omega_{(n)} u = 0$ for all $n \geq 2$ whenever $u \in R$ and $\omega_{(1)} u = u$.

When $(U, Y, \mathbf{1}, R)$ is an enhanced VOA, we call $(U, Y, \mathbf{1}, \omega)$ the *underlying VOA*. The element ω is called the *Virasoro element* of $(U, Y, \mathbf{1}, R)$.

In many instances, the vertex Lie algebra R will be of the form $R = [\Omega]$ (recall §1.4) for some (finite) subset $\Omega \subset U$, and in such a case we may write $(U, Y, \mathbf{1}, \Omega)$ in place of $(U, Y, \mathbf{1}, [\Omega])$ since there is no loss of information. We then regard $(U, Y, \mathbf{1}, \Omega)$ and $(U, Y, \mathbf{1}, \Omega')$ as *identical* enhanced vertex algebras just when $[\Omega] = [\Omega']$; that is, when Ω and Ω' generate the same vertex Lie subalgebra of $\operatorname{Sing}(U, Y, \mathbf{1})$. Also, we will write (U, Ω) or even U in place of $(U, Y, \mathbf{1}, \Omega)$ when no confusion will arise. When $U = (U, Y, \mathbf{1}, R)$ is an enhanced vertex algebra we say that R determines the *conformal structure* on U. (Then there is a convenient coincidence with the terminology of [Kac98], where the objects we refer to as vertex Lie algebras are called *conformal algebras*.) The automorphism group of an enhanced vertex algebra (U, Ω) is the subgroup of the group of vertex algebra automorphisms of U that fixes each element of Ω. In practice, we may write $\operatorname{Aut}(U, \Omega)$ in order to emphasize this. A pair (M, Y^M) is a module over an enhanced VOA $(U, Y, \mathbf{1}, \Omega)$ just when it is a module over $(U, Y, \mathbf{1}, \omega)$. Other notions associated to VOAs (such as rank, rationality, simplicity, &c.) carry over directly to enhanced VOAs in a similar way, via the underlying VOA. A morphism of enhanced vertex algebras $(U_1, \Omega_1) \to (U_2, \Omega_2)$ is a morphism of the underlying VOAs $(U_1, \omega_1) \to (U_2, \omega_2)$ that restricts to a morphism of vertex Lie algebras $[\Omega_1] \to [\Omega_2]$.

For $U = (U, Y, \mathbf{1}, \Omega)$ an enhanced vertex algebra, the set Ω is called a *conformal generating set* for U, and the elements of Ω are called *conformal generators*. Given a conformal generating set Ω for an enhanced vertex algebra U, we say that Ω has *defect* d if d is the minimal non-negative integer such that

$$\omega^1_{(k)} \omega^2 \in \operatorname{Span}\{\mathbf{1}, T^m \nu \mid m \geq 0,\ \nu \in \Omega\} \tag{2.0.5}$$

for all $k \geq d$ and all $\omega^1, \omega^2 \in \Omega$. (The operator T here is the translation operator, defined as in §1.4.) If Ω has defect d then the subspace of $\operatorname{End}(U)$ spanned by the $\nu_{(n)}$ for $\nu \in \Omega \cup \{\mathbf{1}\}$ and $n \in \mathbb{Z}$

$$\operatorname{Span}\{\operatorname{Id}_U, \nu_{(n)} \mid \nu \in \Omega, n \in \mathbb{Z}\} \subset \operatorname{End}(U) \tag{2.0.6}$$

is closed under the d^{th}-order bracket $[\cdot \times_d \cdot]$ (see §1.3). In particular, the case that Ω has defect 0 is just the case that the vertex Lie algebra $[\Omega]$ generated by Ω coincides with the $\mathbb{F}[T]$ module generated by $\Omega \cup \{\mathbf{1}\}$.

Example. Any VOA $(U, Y, \mathbf{1}, \omega)$ furnishes an enhanced VOA (with conformal generating set of defect 0) when we set $\Omega = \{\omega\}$.

Definition. Let $(U, \Omega) = (U, Y, \mathbf{1}, \Omega)$ be an enhanced vertex algebra and set $\mathcal{A}(\Omega) = \mathrm{Lie}([\Omega])$, so that $\mathcal{A}(\Omega)$ is the local Lie algebra of the vertex Lie subalgebra of $\mathrm{Sing}(U, Y, \mathbf{1})$ generated by Ω. We call $\mathcal{A}(\Omega)$ the *local algebra* associated to (U, Ω). If (U, Ω) is an enhanced VOA, then we regard $\mathcal{A}(\Omega)$ as a $\mathbb{Q}$-graded Lie algebra.

For $\mathcal{L}$ a Lie algebra with $\mathbb{Q}$-grading, we say that an enhanced VOA (U, Ω) admits a *conformal structure of type* $\mathcal{L}$, or simply an *$\mathcal{L}$–structure*, if there is a vertex Lie subalgebra $R < [\Omega]$ such that $\mathrm{Lie}(R)$ is isomorphic to $\mathcal{L}$, as graded Lie algebras. Usually we are interested in the case that the choice of vertex Lie subalgebra R here is unique, in which case we say the enhanced VOA (U, Ω) admits a *unique $\mathcal{L}$–structure*.

Example. In [Dun07] we studied an $N = 1$ VOA whose full automorphism group is Conway's largest sporadic group. In the present setting, an $N = 1$ VOA is an enhanced VOA which admits a conformal generating set (of defect 0) consisting of two elements $\Omega = \{\omega, \tau\}$ such that the associated local algebra $\mathcal{A}(\Omega)$ is a copy of the $N = 1$ Virasoro superalgebra. (Indeed, the vector space underlying an $N = 1$ VOA must be a superspace with non-trivial odd part.)

Another case of $\mathcal{L}$–structure that we would like to single out is the case that $\mathcal{L}$ is the (purely even) Lie algebra spanned by symbols J_m, L_m, and $\mathbf{c}$, for $m \in \mathbb{Z}$, and subject to the following relations

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\mathbf{c}, \quad [L_m, \mathbf{c}] = 0, \\ &[L_m, J_n] = -nJ_{m+n}, \quad [J_m, J_n] = -m\delta_{m+n,0}\mathbf{c}, \quad [J_m, \mathbf{c}] = 0. \end{aligned} \tag{2.0.7}$$

We call this algebra the $U(1)$ *Virasoro algebra*. The grading is such that J_m, and L_m have degree $-m$, and $\mathbf{c}$ has degree 0. The subalgebra generated by the L_m is a copy of the Virasoro algebra, and the subalgebra generated by the J_m is a copy of the rank 1 Heisenberg algebra. A representation of the $U(1)$ Virasoro algebra is said to have *rank c* if the central element $\mathbf{c}$ acts as multiplication by c, for some $c \in \mathbb{F}$.

Definition. We say that an enhanced VOA $(U, Y, \mathbf{1}, \Omega)$ is an *enhanced* $U(1)$*–VOA* if there is a unique (up to sign) $\jmath \in [\Omega]$ such that the Fourier components of the operators $Y(\omega, z) = \sum L(m)z^{-m-2}$ and $Y(\jmath, z) = \sum J(m)z^{-m-1}$ furnish a representation of the $U(1)$ Virasoro algebra (2.0.7) under the assignment $L_m \mapsto L(m)$, $J_m \mapsto J(m)$.

Note that for an enhanced $U(1)$–VOA the element $\omega_\alpha = \omega + \alpha T\jmath$ may render $(U, Y, \mathbf{1}, \omega_\alpha)$ a VOA for many choices of $\alpha \in \mathbb{F}$. On the other hand, there is only one choice ($\alpha = 0$) for which $(\omega_\alpha)_{(2)}\jmath = 0$. If $(U, Y, \mathbf{1}, \Omega)$ is an enhanced $U(1)$–VOA and $[\Omega] = [\omega, \jmath]$ then we may call $(U, Y, \mathbf{1}, \Omega)$ simply a $U(1)$*–VOA*. In §6 we will see that $U(1)$–VOAs (and enhanced $U(1)$–VOAs) admit a richer character theory than do ordinary VOAs.

3 Clifford algebras

The construction of VOAs that we will use arises from certain infinite dimensional Clifford algebra modules. In this section we recall some basic properties of Clifford algebras, and also a construction of modules over finite dimensional Clifford algebras using even binary linear codes, which is just a slight generalization of the method for doubly even codes used in [Dun07]. We discuss briefly the group $Spin_{2N}$ in §3.2, and in §3.4 we recall the construction of VOA module structure on modules over certain infinite dimensional Clifford algebras. In §3.5 we review the Hermitian structure that arises naturally on these objects given the existence of a suitable Hermitian form.

3.1 Clifford algebra structure

Recall that $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. We suppose that $\mathfrak{u}$ is an $\mathbb{F}$-vector space of even dimension with non-degenerate symmetric bilinear form $\langle\cdot\,,\cdot\rangle$, and in the case that $\mathbb{F}=\mathbb{R}$ we will assume this form to be positive definite.

We write $\mathrm{Cliff}(\mathfrak{u})$ for the *Clifford algebra over* $\mathbb{F}$ *generated by* $\mathfrak{u}$. More precisely, we set $\mathrm{Cliff}(\mathfrak{u}) = T(\mathfrak{u})/I(\mathfrak{u})$ where $T(\mathfrak{u})$ is the tensor algebra of $\mathfrak{u}$ over $\mathbb{F}$ with unit denoted $\mathbf{1}$, and $I(\mathfrak{u})$ is the ideal of $T(\mathfrak{u})$ generated by all expressions of the form $u\otimes u+\langle u,u\rangle\mathbf{1}$ for $u\in\mathfrak{u}$. The natural algebra structure on $T(\mathfrak{u})$ induces an associative algebra structure on $\mathrm{Cliff}(\mathfrak{u})$. The vector space $\mathfrak{u}$ embeds in $\mathrm{Cliff}(\mathfrak{u})$, and when it is convenient we identify $\mathfrak{u}$ with its image in $\mathrm{Cliff}(\mathfrak{u})$. We also write a in place of $a\mathbf{1}+I(\mathfrak{u})\in\mathrm{Cliff}(\mathfrak{u})$ for $a\in\mathbb{F}$ when no confusion will arise. For $u\in\mathfrak{u}$ we have the relation $u^2=-|u|^2$ in $\mathrm{Cliff}(\mathfrak{u})$. Polarization of this identity yields $uv+vu=-2\langle u,v\rangle$ for $u,v\in\mathfrak{u}$.

The linear transformation on $\mathfrak{u}$ which is -1 times the identity map lifts naturally to $T(\mathfrak{u})$ and preserves $I(\mathfrak{u})$, and hence induces an involution on $\mathrm{Cliff}(\mathfrak{u})$ which we denote by θ. The map θ is often referred to as the *parity involution*. We have $\theta(u_1\cdots u_k)=(-1)^k u_1\cdots u_k$ for $u_1\cdots u_k\in\mathrm{Cliff}(\mathfrak{u})$ with $u_i\in\mathfrak{u}$, and we write $\mathrm{Cliff}(\mathfrak{u})=\mathrm{Cliff}(\mathfrak{u})^0\oplus\mathrm{Cliff}(\mathfrak{u})^1$ for the decomposition into eigenspaces for θ. Define a bilinear form on $\mathrm{Cliff}(\mathfrak{u})$, denoted $\langle\cdot\,,\cdot\rangle$, by setting $\langle\mathbf{1},\mathbf{1}\rangle=1$, and requiring that for $u\in\mathfrak{u}$, the adjoint of left multiplication by u is left multiplication by $-u$. Then the restriction of $\langle\cdot\,,\cdot\rangle$ to $\mathfrak{u}$ agrees with the original form on $\mathfrak{u}$. The *main anti-automorphism* of $\mathrm{Cliff}(\mathfrak{u})$ is the map we denote α, which acts by sending $u_1\cdots u_k$ to $u_k\cdots u_1$ for $u_i\in\mathfrak{u}$.

Suppose that $\mathcal{E}=\{e_i\}_{i\in\Sigma}$ is an orthonormal basis for $\mathfrak{u}$, indexed by a finite set Σ, and suppose that Σ is equipped with some ordering. For $I=\{i_1,\ldots,i_k\}\subset\Sigma$ we write e_I for the element $e_{i_1}e_{i_2}\cdots e_{i_k}$ in $\mathrm{Cliff}(\mathfrak{u})$ just when $i_1<\cdots<i_k$. In this way we obtain an element e_I in $\mathrm{Cliff}(\mathfrak{u})$ for any $I\subset\Sigma$. (We set $e_\emptyset=\mathbf{1}$.) This correspondence depends on the choice of ordering, but our discussion will be invariant with respect to this choice. Note that $e_Ie_J=\pm e_{I+J}$ for any $I,J\subset\Sigma$, and the set $\{e_I\mid I\subset\Sigma\}$ furnishes an orthonormal basis for $\mathrm{Cliff}(\mathfrak{u})$.

3.2 Spin groups

Let us write $\mathrm{Cliff}(\mathfrak{u})^\times$ for the group of invertible elements in $\mathrm{Cliff}(\mathfrak{u})$. For $x\in\mathrm{Cliff}(\mathfrak{u})^\times$ and $a\in\mathrm{Cliff}(\mathfrak{u})$, we set $x(a)=xax^{-1}$. We will define the Pinor and Spinor groups associated to $\mathfrak{u}$ slightly differently according as $\mathfrak{u}$ is real or complex: in the case that $\mathfrak{u}$ is real, we define the *Pinor group* $Pin(\mathfrak{u})$ to be the subgroup of $\mathrm{Cliff}(\mathfrak{u})^\times$ comprised of elements x such that $x(\mathfrak{u})\subset\mathfrak{u}$ and $\alpha(x)x=\pm1$;

in the case that $\mathfrak{u}$ is complex we define $Pin(\mathfrak{u})$ to be the set of $x \in \mathrm{Cliff}(\mathfrak{u})^\times$ such that $x(\mathfrak{u}) \subset \mathfrak{u}$ and $\alpha(x)x = 1$. In both cases we define the *Spinor group* by setting $Spin(\mathfrak{u}) = Pin(\mathfrak{u}) \cap \mathrm{Cliff}(\mathfrak{u})^0$.

Let $x \in Pin(\mathfrak{u})$. Then we have $\langle x(u), x(v)\rangle = \langle u, v\rangle$ for $u, v \in \mathfrak{u}$, and thus the map $x \mapsto x(\cdot)$, which has kernel $\pm\mathbf{1}$, realizes the Pinor group as a double cover of $O(\mathfrak{u})$. (If $u \in \mathfrak{u}$ and $\langle u, u\rangle = 1$, then $u(\cdot)$ is the orthogonal transformation of $\mathfrak{u}$ which is minus the reflection in the hyperplane orthogonal to u.) The image of $Spin(\mathfrak{u})$ under the map $x \mapsto x(\cdot)$ is just the group $SO(\mathfrak{u})$.

In the case that $\mathfrak{u}$ is real with definite bilinear form, we have $\alpha(x)x = 1$ for all $x \in Spin(\mathfrak{u})$, and the group $Spin(\mathfrak{u})$ is generated (as a subgroup of $\mathrm{Cliff}(\mathfrak{u})^\times$) by the expressions $\exp(\lambda e_i e_j)$ for $\lambda \in \mathbb{R}$ and $\{e_i\}$ an orthonormal basis of $\mathfrak{u}$. The Spinor group of the complexified space ${}_{\mathbb{C}}\mathfrak{u}$ is then generated by the $\exp(\lambda e_i e_j)$ for $\lambda \in \mathbb{C}$.

3.3 Clifford algebra modules

We obtain an $\mathbb{F}_2^\Sigma$-grading on $\mathrm{Cliff}(\mathfrak{u})$ by decreeing that for $I \subset \Sigma$, the homogeneous subspace of $\mathrm{Cliff}(\mathfrak{u})$ with degree I is just the $\mathbb{F}$-span of the vector e_I.

$$\mathrm{Cliff}(\mathfrak{u}) = \bigoplus_{I \subset \Sigma} \mathrm{Cliff}(\mathfrak{u})^I, \quad \mathrm{Cliff}(\mathfrak{u})^I = \mathbb{F}e_I. \tag{3.3.1}$$

Since this grading depends on the choice of orthonormal basis $\mathcal{E}$, we will refer to it as the $\mathbb{F}_2^{\mathcal{E}}$-grading, and we refer to the homogeneous elements ae_I for $a \in \mathbb{F}$ as $\mathbb{F}_2^{\mathcal{E}}$-homogeneous elements. A given subset of $\mathrm{Cliff}(\mathfrak{u})$ is called $\mathbb{F}_2^{\mathcal{E}}$-homogeneous if all of its elements are $\mathbb{F}_2^{\mathcal{E}}$-homogeneous.

Suppose that E is an $\mathbb{F}_2^{\mathcal{E}}$-homogeneous subgroup of $Spin(\mathfrak{u})$ such that the natural map $E \to \mathbb{F}_2^\Sigma$ is injective. Then E is a union of elements of the form $\pm e_C$ or $\pm\mathbf{i}e_C$ for $C \subset \Sigma$, and $-\mathbf{1} \notin E$, and the image of E in $\mathbb{F}_2^\Sigma$ is a binary linear code on Σ. For E such a subgroup of $Spin(\mathfrak{u})$, we write $\mathcal{C}(E)$ for the associated code, and we call E an $\mathbb{F}_2^{\mathcal{E}}$-homogeneous lift of $\mathcal{C}(E)$.

Suppose now that $\mathcal{C}(E)$ is a self-dual even code. (In the case that $\mathbb{F} = \mathbb{R}$ this forces $\mathcal{C}(E)$ to be a doubly even code, and this in turn forces $\dim(\mathfrak{u})$ to be a multiple of eight.) Then we write $\mathrm{CM}(\mathfrak{u})_E$ for the $\mathrm{Cliff}(\mathfrak{u})$-module defined by $\mathrm{CM}(\mathfrak{u})_E = \mathrm{Cliff}(\mathfrak{u}) \otimes_{\mathbb{F}E} \mathbb{F}_1$ where $\mathbb{F}_1$ denotes a trivial E-module. Let us set $1_E = \mathbf{1} \otimes 1 \in \mathrm{CM}(\mathfrak{u})_E$. Then $\mathrm{CM}(\mathfrak{u})_E$ admits a bilinear form defined so that $\langle 1_E, 1_E\rangle = 1$, and the adjoint to left multiplication by $u \in \mathfrak{u} \hookrightarrow \mathrm{Cliff}(\mathfrak{u})$ is left multiplication by $-u$.

Proposition 3.1. *The* $\mathrm{Cliff}(\mathfrak{u})$*-module* $\mathrm{CM}(\mathfrak{u})_E$ *is irreducible, and a vector-space basis for* $\mathrm{CM}(\mathfrak{u})_E$ *is naturally indexed by the elements of the co-code* $\mathcal{C}(E)^*$.

Proof. We have $e_{S+C}1 = \pm e_S 1$ for any $S \subset \Sigma$ when $C \in \mathcal{C}(E)$. This shows that a basis for $\mathrm{CM}(\mathfrak{u})_E$ is indexed by the elements of the co-code $\mathcal{C}(E)^* = \mathbb{F}_2^\Sigma/\mathcal{C}(E)$, and it follows that the irreducible submodules of $\mathrm{CM}(\mathfrak{u})_E$ are indexed by the cosets of $\mathcal{C}(E)$ in its dual code $\mathcal{C}(E)^\circ$ (see §0.2). Since $\mathcal{C}(E)$ is assumed to be self-dual, $\mathrm{CM}(\mathfrak{u})_E$ is irreducible. $\square$

Note that the vector $1_E \in \mathrm{CM}(\mathfrak{u})_E$ is such that $g1_E = 1_E$ for all $g \in E$, and $\mathrm{CM}(\mathfrak{u})_E$ is spanned by the $a1_E$ for $a \in \mathrm{Cliff}(\mathfrak{u})$.

3.4 Clifford module VOAs

In this section we review the construction of VOA structure on modules over a certain infinite dimensional Clifford algebra associated to $\mathfrak{u}$. The construction is quite standard and one may refer to [FFR91] for example, for all the details. Our setup is somewhat different from that in [FFR91] in that we also wish to be able to handle the case that $\mathbb{F} = \mathbb{R}$, and we must therefore use an alternative construction of the canonically twisted VOA module since a polarization of $\mathfrak{u}$ does not exist in this case. On the other hand, all one requires is an irreducible module over the (finite dimensional) Clifford algebra $\mathrm{Cliff}(\mathfrak{u})$, and the arguments of [FFR91] then go through with only cosmetic changes.

Let $\hat{\mathfrak{u}}$ and $\hat{\mathfrak{u}}_\theta$ denote the infinite dimensional inner product spaces described as follows.

$$\hat{\mathfrak{u}} = \coprod_{m\in\mathbb{Z}} \mathfrak{u}\otimes t^{m+1/2}, \quad \hat{\mathfrak{u}}_\theta = \coprod_{m\in\mathbb{Z}} \mathfrak{u}\otimes t^{m}, \tag{3.4.1}$$

$$\langle u\otimes t^r, v\otimes t^s\rangle = \langle u,v\rangle\delta_{r+s,0}, \quad \text{for } u,v\in\mathfrak{u} \text{ and } r,s\in\tfrac{1}{2}\mathbb{Z}. \tag{3.4.2}$$

We write $u(r)$ for $u\otimes t^r$ when $u\in\mathfrak{u}$ and $r\in\frac{1}{2}\mathbb{Z}$. We consider the Clifford algebras $\mathrm{Cliff}(\hat{\mathfrak{u}})$ and $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$. The inclusion of $\mathfrak{u}$ in $\hat{\mathfrak{u}}_\theta$ given by $u\mapsto u(0)$ induces an embedding of algebras $\mathrm{Cliff}(\mathfrak{u})\hookrightarrow\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$. For $S=(i_1,\dots,i_k)$ an ordered subset of Σ we write $e_S(r)$ for the element $e_{i_1}(r)\cdots e_{i_k}(r)$, which lies in $\mathrm{Cliff}(\hat{\mathfrak{u}})$ or $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$ according as r is in $\mathbb{Z}+\frac{1}{2}$ or $\mathbb{Z}$. With this notation $e_S(0)$ coincides with the image of e_S under the embedding $\mathrm{Cliff}(\mathfrak{u})\hookrightarrow\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$.

Let $\mathcal{C}$ be an even self-dual code on Σ, and let $E<\mathit{Spin}(\mathfrak{u})$ be an $\mathbb{F}_2^{\mathcal{E}}$-homogeneous lift of $\mathcal{C}$ (see §3.3). Note that in the case $\mathbb{F}=\mathbb{R}$ this forces $\mathcal{C}$ to be a doubly even code, and that in turn forces $\mathfrak{u}$ to have dimension divisible by 8. We write $\mathcal{B}(\hat{\mathfrak{u}})$ for the subalgebra of $\mathrm{Cliff}(\hat{\mathfrak{u}})$ generated by the $u(m+\frac{1}{2})$ for $u\in\mathfrak{u}$ and $m\in\mathbb{Z}_{\geq 0}$. We write $\mathcal{B}(\hat{\mathfrak{u}}_\theta)_E$ for the subalgebra of $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$ generated by $E\subset\mathrm{Cliff}(\mathfrak{u})$, and the $u(m)$ for $u\in\mathfrak{u}$ and $m\in\mathbb{Z}_{>0}$. Let $\mathbb{F}_1$ denote a one-dimensional module for either $\mathcal{B}(\hat{\mathfrak{u}})$ or $\mathcal{B}(\hat{\mathfrak{u}}_\theta)_E$, spanned by a vector 1_E, such that $u(r)1_E=0$ whenever $r\in\frac{1}{2}\mathbb{Z}_{>0}$, and such that $g(0)1_E=1_E$ for $g\in E$. We write $A(\mathfrak{u})$ (respectively $A(\mathfrak{u})_{\theta,E}$) for the $\mathrm{Cliff}(\hat{\mathfrak{u}})$-module (respectively $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$-module) induced from the $\mathcal{B}(\hat{\mathfrak{u}})$-module structure (respectively $\mathcal{B}(\hat{\mathfrak{u}}_\theta)_E$-module structure) on $\mathbb{F}_1$.

$$A(\mathfrak{u}) = \mathrm{Cliff}(\hat{\mathfrak{u}})\otimes_{\mathcal{B}(\hat{\mathfrak{u}})}\mathbb{F}_1, \qquad A(\mathfrak{u})_{\theta,E} = \mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)\otimes_{\mathcal{B}(\hat{\mathfrak{u}}_\theta)_E}\mathbb{F}_1. \tag{3.4.3}$$

We write $\mathbf{1}$ for the vector $1\otimes 1_E$ in $A(\mathfrak{u})$, and we write $\mathbf{1}_\theta$ or $\mathbf{1}_E$ for the vector $1\otimes 1_E$ in $A(\mathfrak{u})_{\theta,E}$.

The space $A(\mathfrak{u})$ supports a structure of VOA. In order to define the vertex operators we require the notion of fermionic normal ordering for elements in $\mathrm{Cliff}(\hat{\mathfrak{u}})$ and $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$. The fermionic normal ordering on $\mathrm{Cliff}(\hat{\mathfrak{u}})$ is the multi-linear operator defined so that for $u_i\in\mathfrak{u}$ and $r_i\in\mathbb{Z}+\frac{1}{2}$ we have

$$:u_1(r_1)\cdots u_k(r_k):=\ \mathrm{sgn}(\sigma)u_{\sigma1}(r_{\sigma1})\cdots u_{\sigma k}(r_{\sigma k}) \tag{3.4.4}$$

where σ is any permutation of the index set $\{1,\dots,k\}$ such that $r_{\sigma1}\leq\cdots\leq r_{\sigma k}$. For elements in $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$ the fermionic normal ordering is defined in steps by first setting

$$:u_1(0)\cdots u_k(0):=\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)u_{\sigma1}(0)\cdots u_{\sigma k}(0) \tag{3.4.5}$$

for $u_i \in \mathfrak{u}$. Then in the situation that $n_i \in \mathbb{Z}$ are such that $n_i \leq n_{i+1}$ for all i, and there are some s and t (with $1 \leq s \leq t \leq k$) such that $n_j = 0$ for $s \leq j \leq t$, we set

$$\begin{aligned} &:u_1(n_1)\cdots u_k(n_k): \\ &= u_1(n_1)\cdots u_{s-1}(n_{s-1}):u_s(0)\cdots u_t(0):u_{t+1}(n_{t+1})\cdots u_k(n_k) \end{aligned} \tag{3.4.6}$$

Finally, for arbitrary $n_i \in \mathbb{Z}$ we set

$$:u_1(n_1)\cdots u_k(n_k):= \operatorname{sgn}(\sigma):u_{\sigma 1}(n_{\sigma 1})\cdots u_{\sigma k}(n_{\sigma k}): \tag{3.4.7}$$

where σ is again any permutation of the index set $\{1,\ldots,k\}$ such that $n_{\sigma 1} \leq \cdots \leq n_{\sigma k}$, and we extend the definition multilinearly to $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$.

For $u \in \mathfrak{u}$ we now define the generating function, denoted $u(z)$, of operators on $A(\mathfrak{u})_\Theta = A(\mathfrak{u}) \oplus A(\mathfrak{u})_\theta$ by setting

$$u(z) = \sum_{r \in \frac{1}{2}\mathbb{Z}} u(r) z^{-r-1/2} \tag{3.4.8}$$

Note that $u(r)$ acts as 0 on $A(\mathfrak{u})$ if $r \in \mathbb{Z}$, and acts as 0 on $A(\mathfrak{u})_\theta$ if $r \in \mathbb{Z} + \frac{1}{2}$. To an element $a \in A(\mathfrak{u})$ of the form $a = u_1(-m_1 - \frac{1}{2})\cdots u_k(-m_k - \frac{1}{2})\mathbf{1}$ for $u_i \in \mathfrak{u}$ and $m_i \in \mathbb{Z}_{\geq 0}$, we associate the operator valued power series $\overline{Y}(a,z)$, given by

$$\overline{Y}(a,z) = :D_z^{(m_1)} u_{i_1}(z)\cdots D_z^{(m_k)} u_{i_k}(z): \tag{3.4.9}$$

We define the vertex operator correspondence

$$Y(\cdot\,,z) : A(\mathfrak{u}) \otimes A(\mathfrak{u})_\Theta \to A(\mathfrak{u})_\Theta((z^{1/2})) \tag{3.4.10}$$

by setting $Y(a,z)b = \overline{Y}(a,z)b$ when $b \in A(\mathfrak{u})$, and by setting $Y(a,z)b = \overline{Y}(e^{\Delta_z}a,z)b$ when $b \in A(\mathfrak{u})_\theta$, where Δ_z is the expression defined by

$$\Delta_z = -\frac{1}{4}\sum_i \sum_{m,n \in \mathbb{Z}_{\geq 0}} C_{mn} e_i(m+\tfrac{1}{2}) e_i(n+\tfrac{1}{2}) z^{-m-n-1} \tag{3.4.11}$$

$$C_{mn} = \frac{1}{2}\frac{(m-n)}{m+n+1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n} \tag{3.4.12}$$

Set $\omega = \frac{1}{4}\sum_i e_i(-\frac{1}{2})e_i(-\frac{3}{2})\mathbf{1} \in A(\mathfrak{u})_2$. Then one has the following

Theorem 3.2 ([FFR91])**.** *The map Y defines a structure of self-dual VOA of rank N on $A(\mathfrak{u})$ when restricted to $A(\mathfrak{u}) \otimes A(\mathfrak{u})$, and the Virasoro element is given by ω. The map Y defines a structure of θ-twisted $A(\mathfrak{u})$-module on $A(\mathfrak{u})_\theta$ when restricted to $A(\mathfrak{u}) \otimes A(\mathfrak{u})_\theta$.*

Observe that $A(\mathfrak{u})_2$ is spanned by vectors of the form $e_I(-\frac{1}{2})\mathbf{1}$ for $I \subset \Sigma$ with $|I| = 4$, and by the $e_i(-\frac{3}{2})e_j(-\frac{1}{2})\mathbf{1}$ with $i,j \in \Sigma$.

Essentially all we need to know about the expressions $Y(a,z)b$ for $b \in A(\mathfrak{u})_\theta$ is contained in the following

Proposition 3.3. *Let $b \in A(\mathfrak{u})_\theta$.*

1. *If $a = \mathbf{1} \in A(\mathfrak{u})_0$ then $Y(a,z)b = b$.*
2. *If $a \in A(\mathfrak{u})_1$ then $\Delta_z a = 0$ so that $Y(a,z)b = \overline{Y}(a,z)b$.*
3. *If $a \in A(\mathfrak{u})_2$ and $a \in \operatorname{Span}\left\{e_I(-\frac{1}{2})\mathbf{1}, e_i(-\frac{3}{2})e_j(-\frac{1}{2})\mathbf{1} \mid i \neq j\right\}$ then $\Delta_z a = 0$ and $Y(a,z)b = \overline{Y}(a,z)b$.*
4. *For $a = e_i(-\frac{1}{2})e_i(-\frac{3}{2})\mathbf{1}$ we have $\Delta_z a = \frac{1}{4}z^{-2}$ and $\Delta_z^2 a = 0$ so that $Y(a,z)b = \overline{Y}(a,z)b + \frac{1}{4}bz^{-2}$ in this case.*

As a corollary of Proposition 3.3 we have that

$$Y(\omega,z)\mathbf{1}_\theta = \frac{N}{8}\mathbf{1}_\theta z^{-2} \tag{3.4.13}$$

and consequently the $L(0)$-grading on $A(\mathfrak{u})_\Theta$ is given by

$$A(\mathfrak{u}) = \coprod_{n\in\frac{1}{2}\mathbb{Z}_{\geq 0}} A(\mathfrak{u})_n, \quad A(\mathfrak{u})_\theta = \coprod_{n\in\frac{1}{2}\mathbb{Z}_{\geq 0}} (A(\mathfrak{u})_\theta)_{n+N/8}. \tag{3.4.14}$$

Given a specific choice of E, the embedding of $\mathrm{Cliff}(\mathfrak{u})$ in $\mathrm{Cliff}(\hat{\mathfrak{u}}_\theta)$ gives rise to an isomorphism of $\mathrm{CM}(\mathfrak{u})_E$ with $(A(\mathfrak{u})_{\theta,E})_{N/8}$, and it will be convenient to consider these spaces as identified.

The group $Spin(\mathfrak{u})$ acts naturally on $A(\mathfrak{u})_\Theta$, and this action is generated by the exponentials of the operators $x_{(0)}$ for $x \in A(\mathfrak{u})_1$. In particular, any $a \in Spin(\mathfrak{u}) \subset \mathrm{Cliff}(\mathfrak{u})$ may be regarded as a VOA automorphism of $A(\mathfrak{u})$, and as an equivariant linear isomorphism of the $A(\mathfrak{u})$-module $A(\mathfrak{u})_{\theta,X}$. Recall the element $\mathfrak{z} = e_\Sigma$ of §3.2. When constructing a realization of the twisted module $A(\mathfrak{u})_\theta$ we will always choose E so that $e_\Sigma \in E$, and thus $\mathfrak{z}$ will be the unique preimage in $Spin(\mathfrak{u})$ of $-\mathrm{Id}_\mathfrak{u}$ that fixes the vector $\mathbf{1}_E$ in $A(\mathfrak{u})_{\theta,E}$. Equivalently, $\mathfrak{z}$ is the unique element of $Spin(\mathfrak{u})$ whose action on $A(\mathfrak{u})_\Theta$ coincides with that of the parity involution θ.

3.5 Hermitian structure

We will now take special interest in the case that $\mathbb{F} = \mathbb{C}$ and $\mathfrak{u}$ is of the form $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$ for $\mathfrak{a}$ some complex vector space with non-degenerate Hermitian form, and $\mathfrak{a}^*$ the dual space to $\mathfrak{a}$. We shall denote the Hermitian form by $(\cdot\,,\cdot)$, and our convention will be that a Hermitian form be anti-linear in the second slot. In this instance we insist that the bilinear form $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{u}$ be $1/2$ the symmetric $\mathbb{C}$-bilinear form on $\mathfrak{u}$ induced by the natural pairing between $\mathfrak{a}$ and $\mathfrak{a}^*$. That is, we set $\langle a, f\rangle = \langle f, a\rangle = \frac{1}{2}f(a)$ for $a \in \mathfrak{a}$ and $f \in \mathfrak{a}^*$. We suppose that Δ is some set with cardinality N, and that $\{a_i\}_{i\in\Delta}$ is a basis for $\mathfrak{a}$ satisfying $(a_i, a_j) = \delta_{ij}$ for $i, j \in \Delta$. We suppose also that $\{a_i^*\}_{i\in\Delta}$ is the dual basis for $\mathfrak{a}^*$, so that $\langle a_i, a_j^*\rangle = \langle a_j^*, a_i\rangle = \frac{1}{2}\delta_{ij}$. With this convention we have $a_i a_j^* + a_j^* a_i + \delta_{ij} = 0$ in $\mathrm{Cliff}(\mathfrak{u})$.

The Hermitian form on $\mathfrak{a}$ entails an anti-involution ϑ defined on $\mathfrak{u}$ in such a way that ϑ interchanges αa_i with $\bar{\alpha} a_i^*$ for $\alpha \in \mathbb{C}$ and $i \in \Delta$. We then have the identity $(u, v) = 2\langle u, \vartheta v\rangle$ for

$u, v \in \mathfrak{u}$, and our definition of ϑ is invariant with respect to the choice of orthonormal basis for $\mathfrak{a}$. The fixed points under ϑ form a real vector space of dimension $2N$ in $\mathfrak{u}$. We will write ${}_{\mathbb{R}}\mathfrak{u}$ for this space, keeping in mind the dependence upon the choice of Hermitian structure for $\mathfrak{a}$. With such $\mathfrak{u}$ we make the convention that $\mathcal{E}$ shall denote the orthonormal basis for ${}_{\mathbb{R}}\mathfrak{u}$ obtained in the following way. We let Δ' be another copy of the set Δ with the natural identification $\Delta \leftrightarrow \Delta'$ denoted $i \leftrightarrow i'$, and we set $\mathcal{E} = \{e_i, e_{i'}\}_{i\in\Delta}$ where

$$e_i = a_i + a_i^*, \quad e_{i'} = \mathbf{i}(a_i - a_i^*), \tag{3.5.1}$$

for $i \in \Delta$. We let $\Sigma = \Delta \cup \Delta'$ and we agree to order Σ in this case by first choosing an ordering on Δ, and then setting $i < j'$ for all $i, j \in \Delta$.

In our present situation there is a standard choice of group to play the role of E in §3.3. We reserve the notation X for the subgroup of $Spin(\mathfrak{u})$ generated by elements of the form $\mathbf{i}e_ie_{i'}$ for $i \in \Sigma$, and we may then take $E = X$ in §3.3, so as to obtain the $\mathrm{Cliff}(\mathfrak{u})$-module $\mathrm{CM}(\mathfrak{u})_X$. We note here that $\mathrm{CM}(\mathfrak{u})_X$ may be characterized as the left-module over $\mathrm{Cliff}(\mathfrak{u})$ spanned by a vector 1_X such that $e_{i'}1_X = \mathbf{i}e_i1_X$ for all $i \in \Delta$, or equivalently, such that $a_i^*1_X = 0$ for all $i \in \Delta$. In particular, $\mathrm{CM}(\mathfrak{u})_X$ may be naturally identified with the space $\bigwedge(\mathfrak{a})$. When no confusion will arise we regard $\bigwedge(\mathfrak{a})$ as a $\mathrm{Cliff}(\mathfrak{u})$ module via this identification. Note also the identity $e_{i_1}\cdots e_{i_k}1_X = a_{i_1}\cdots a_{i_k}1_X$ in $\mathrm{CM}(\mathfrak{u})_X$ for $I = \{i_1, \ldots, i_k\} \subset \Delta$.

4 Linear groups

In this section we define families of enhanced VOAs with symmetry groups related to the linear groups $GL_N(\mathbb{C})$ for N a positive integer. In defining these objects we are laying some of the ground work for the construction of the enhanced VOA for the Rudvalis group in §5.

4.1 Untwisted construction

Let N be a positive integer and let $\mathfrak{a}$ be a complex vector space of dimension N equipped with a positive definite Hermitian form denoted $(\cdot\,,\cdot)$. As in §3.5 we set $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$, we extend the Hermitian form to $\mathfrak{u}$, and equip this space also with the symmetric bilinear form $\langle\cdot\,,\cdot\rangle$ which is just the form arising naturally from the pairing between $\mathfrak{a}$ and $\mathfrak{a}^*$ scaled by a factor of $1/2$.

Let us set $\Omega_U = \{\omega, \jmath\}$ where ω and $\jmath$ are given by

$$\omega = \frac{1}{4}\sum_{i\in\Sigma\cup\Sigma'} e_i(-\tfrac{1}{2})e_i(-\tfrac{3}{2})\mathbf{1} \in A(\mathfrak{u})_2, \tag{4.1.1}$$

$$\jmath = \frac{1}{2}\sum_{i\in\Sigma} e_i(-\tfrac{1}{2})e_{i'}(-\tfrac{1}{2})\mathbf{1} \tag{4.1.2}$$

$$= \mathbf{i}\sum_{i\in\Sigma} a_i^*(-\tfrac{1}{2})a_i(-\tfrac{1}{2})\mathbf{1} \in A(\mathfrak{u})_1, \tag{4.1.3}$$

so that ω is just the usual Virasoro element for $A(\mathfrak{u})$, and $\jmath$ is some element dependent upon the

Hermitian structure on $\mathfrak{a}$. We define operators $L(n)$ and $J(n)$ for $n \in \mathbb{Z}$ by setting

$$Y(\omega, z) = \; L(z) = \sum L(n) z^{-n-2}, \tag{4.1.4}$$

$$Y(\jmath, z) = \; J(z) = \sum J(n) z^{-n-1}. \tag{4.1.5}$$

We then have

Proposition 4.1. *The vertex operators $L(z)$ and $J(z)$ satisfy the following OPEs*

$$L(z)L(w) = \frac{N/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{D_w L(w)}{(z-w)} + \text{reg.} \tag{4.1.6}$$

$$L(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{D_w J(w)}{(z-w)} + \text{reg.} \tag{4.1.7}$$

$$J(z)J(w) = -\frac{N}{(z-w)^2} + \text{reg.} \tag{4.1.8}$$

We recalled in §1.3 how the OPE of two vertex operators encodes the commutation relations of their component operators. Proposition 4.1 entails the following

Corollary 4.2. *The operators $L(m)$ and $J(m)$ satisfy*

$$\begin{aligned} &[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0} N \,\mathrm{Id} \\ &[L(m), J(n)] = -nJ(m+n), \quad [J(m), J(n)] = -m\delta_{m+n,0} N \,\mathrm{Id} \end{aligned} \tag{4.1.9}$$

for all $m, n \in \mathbb{Z}$.

The objects $(A(\mathfrak{u}), \Omega_U)$ we have constructed provide our first family of examples of enhanced VOAs beyond ordinary VOAs and $N = 1$ VOAs. In fact they are examples of $U(1)$–VOAs (see §2), as is shown by the following

Proposition 4.3. *For N a positive integer, the quadruple $(A(\mathfrak{u}), Y, \mathbf{1}, \Omega_U)$ is a self-dual $U(1)$–VOA of rank N.*

Proof. We will show that $(A(\mathfrak{u}), \Omega_U)$ is a $U(1)$–VOA since the other claims have been verified already. From Proposition 4.1 we see that Ω_U has defect 0, so that the vertex Lie algebra generated by Ω_U is just $[\Omega_U] = \mathrm{Span}\{\mathbf{1}, T^k\jmath, T^k\omega \mid k \in \mathbb{Z}\}$. We have seen that the element $\omega \in \Omega_U$ is such that $(A(\mathfrak{u}), Y, \mathbf{1}, \omega)$ is a VOA. From the explicit spanning set given we see that the only other Virasoro elements in $[\Omega_U]$ are of the form $\omega_\alpha = \omega + \alpha T\jmath$ for some $\alpha \in \mathbb{C}$. Using Corollary 4.2 we check that the Fourier coefficients of $Y(\omega_\alpha, z) = L_\alpha(m)z^{-m-2}$ furnish a representation of the Virasoro algebra of rank $N(1+12\alpha^2)$, and we have $L_\alpha(n) = L(n) + \alpha(-n-1)J(n)$ where $Y(\omega, z) = \sum L(m)z^{-m-2}$ and $Y(\jmath, z) = \sum J(m)z^{-m-1}$. In particular, $L_\alpha(-1) = L(-1) = T$ for all α, and $L_\alpha(0) = L(0) - \alpha J(0)$. Thus if $u \in [\Omega_U]$ and $L_\alpha(0)u = u$ then $u \in \mathrm{Span}\{\jmath\}$. Now we compute $L_\alpha(1)\jmath = \alpha(-2)J(1)\jmath = 2\alpha N$, and it follows that $(A(\mathfrak{u}), Y, \mathbf{1}, \Omega_U)$ is an enhanced VOA with the Virasoro element given by $\omega = \omega_0$. From Corollary 4.2 we see that $\jmath$ is the unique (up to sign) choice of element in $[\Omega_U] \cap (A(\mathfrak{u}))_1$ for which the component operators of $Y(\jmath, z)$ satisfy the required relations (2.0.7). ∎

The proof of Theorem 4.3 shows also that $(A(\mathfrak{u})_{\bar{0}}, Y, \mathbf{1}, \Omega_U)$ is a $U(1)$–VOA.

We would like to compute the automorphism group of $(A(\mathfrak{u}), \Omega_U)$. First we will compute the automorphism groups of the underlying VOA, and of its even subVOA.

Proposition 4.4. *Let $N > 4$. Then the automorphism group of $A(\mathfrak{u})$ is $O(\mathfrak{u})$, and the automorphism group of $A(\mathfrak{u})_{\bar{0}}$ is $O(\mathfrak{u})/\langle \pm \mathrm{Id}\rangle$.*

Proof. Let $\mathfrak{g} = \{x_0 \mid x \in A(\mathfrak{u})_1\} \subset \mathrm{End}(A(\mathfrak{u}))$. Then $\mathfrak{g}$ is a simple complex Lie algebra of type D_N. We set $G = \mathrm{Aut}(A(\mathfrak{u}))$ and $G_0 = \mathrm{Aut}(A(\mathfrak{u})_{\bar{0}})$, and we write S for the subgroup of G generated by the exponentials e^X for $X \in \mathfrak{g}$. Note that any automorphism of $A(\mathfrak{u})$ must preserve the even subspace $A(\mathfrak{u})_{\bar{0}}$, so there is a natural map $\phi : G \to G_0$. The kernel of this map is generated by the canonical automorphism of U; viz. the automorphism that fixes the even subspace and negates the odd subspace. In particular, $\ker(\phi)$ is contained in S. Note also that there is a natural isomorphism $G_0 \cong \mathrm{Aut}(\mathfrak{g})$ since $A(\mathfrak{u})_{\bar{0}}$ is generated by its subspace of elements of degree 1.

$$A(\mathfrak{u})_{\bar{0}} = \mathrm{Span}\left\{u^1_{-n_1} \cdots u^k_{-n_k}\mathbf{1} \mid u^i \in A(\mathfrak{u})_1,\, n_1 \geq \cdots \geq n_k > 0\right\} \tag{4.1.10}$$

Now the image of $\phi(S)$ in $\mathrm{Aut}(\mathfrak{g})$ is the subgroup $\mathrm{Inn}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$. At least for $N > 4$ then, $\phi(S)$ has index two in G_0 since $\mathfrak{g}$ is of type D_N, so $G_0 = \phi(S) \cup \bar{x}\phi(S)$ for some $\bar{x} \in G_0 \setminus \phi(S)$.

Observe now that $A(\mathfrak{u})$ is generated by its subspace of degree 1/2,

$$A(\mathfrak{u}) = \mathrm{Span}\left\{u^1_{-n_1} \cdots u^k_{-n_k}\mathbf{1} \mid u^i \in A(\mathfrak{u})_{1/2},\, n_1 \geq \cdots \geq n_k > 0\right\} \tag{4.1.11}$$

which is naturally identified with $\mathfrak{u}$ under the correspondence $u \leftrightarrow u(-\frac{1}{2})\mathbf{1}$ for $u \in \mathfrak{u}$. Further, this correspondence identifies the bilinear form on $\mathfrak{u}$ with the restriction to $A(\mathfrak{u})_{1/2}$ of the invariant form on $A(\mathfrak{u})$. We see then that any automorphism of $A(\mathfrak{u})$ induces an element of $O(\mathfrak{u})$ by restriction to the degree 1/2 subspace, and conversely, any element of $O(\mathfrak{u})$ extends to an automorphism of $A(\mathfrak{u})$, and thus we may identify $\mathrm{Aut}(A(\mathfrak{u}))$ as $O(\mathfrak{u})$. We see also, that the map $\phi : G \to G_0$ is surjective, since $O(\mathfrak{u})$ acts non-trivially on the even subVOA $A(\mathfrak{u})_{\bar{0}}$. The group S must then be $SO(\mathfrak{u})$, and we may regard the $\bar{x}$ in $G_0 = \phi(S) \cup \bar{x}\phi(S)$ as the image under ϕ of an element x say, in $O(\mathfrak{u}) \setminus SO(\mathfrak{u})$. Since the kernel of ϕ is the canonical automorphism group of $A(\mathfrak{u})$ (generated by $-\mathrm{Id}$), we see that G_0 is the group $O(\mathfrak{u})/\langle \pm \mathrm{Id}\rangle$. □

Proposition 4.5. *The automorphism group of $(A(\mathfrak{u}), \Omega_U)$ is $GL(\mathfrak{a})$. The automorphism group of $(A(\mathfrak{u})_{\bar{0}}, \Omega_U)$ is $GL(\mathfrak{a})/\langle \pm \mathrm{Id}\rangle$.*

Proof. Recall that $\jmath = \sum \mathbf{i} a_i^*(-\frac{1}{2})a_i(-\frac{1}{2})\mathbf{1}$. From the previous proposition $\mathrm{Aut}(A(\mathfrak{u}), \{\omega\}) = O(\mathfrak{u})$. The group $\mathrm{Aut}(A(\mathfrak{u}), \Omega_U)$ is just the subgroup of $\mathrm{Aut}(A(\mathfrak{u}))$ that fixes $\jmath$. Consider the automorphism of $A(\mathfrak{u})$ obtained by setting $\theta^{1/2} = \exp(\pi J(0)/2)$. It is the automorphism of $A(\mathfrak{u})$ induced by the orthogonal transformation of $\mathfrak{u}$ which is multiplication by $\mathbf{i}$ on $\mathfrak{a}$, and multiplication by $-\mathbf{i}$ on $\mathfrak{a}^*$. Clearly, any element of $\mathrm{Aut}(A(\mathfrak{u}), \Omega_U)$ commutes with $\theta^{1/2}$. On the other hand, if $g \in O(\mathfrak{u})$ commutes with $\theta^{1/2}$ then g preserves the decomposition $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$, and with respect to the basis $\{a_1, \ldots, a_1^*, \ldots\}$ is represented by a block matrix

$$g \sim \begin{pmatrix} T_g^t & 0 \\ 0 & T_g^{-1} \end{pmatrix} \tag{4.1.12}$$

for some invertible $N\times N$ matrix T_g, where T_g^t denotes the transpose of T_g. Evidently $\mathrm{Aut}(A(\mathfrak{u}),\Omega_U)$ is the centralizer of $\theta^{1/2}$ in $O(\mathfrak{u})$, and this group is all matrices of the form (4.1.12). Since the natural map $\mathrm{Aut}(A(\mathfrak{u}))\to\mathrm{Aut}(A(\mathfrak{u})_{\bar{0}})$ is surjective, the natural map $\mathrm{Aut}(A(\mathfrak{u}),\Omega_U)\to\mathrm{Aut}(A(\mathfrak{u})_{\bar{0}},\Omega_U)$ is also surjective. The kernel in each case is $\pm\,\mathrm{Id}$, and this completes the proof. □

Note that the embedding of $GL(\mathfrak{a})$ in $O(\mathfrak{u})$ given by (4.1.12) is actually an embedding of $GL(\mathfrak{a})$ in $SO(\mathfrak{u})$.

We record some of the results of this section in

Theorem 4.6. *For N a positive integer, the quadruple $(A(\mathfrak{u}),Y,\mathbf{1},\Omega_U)$ is a self-dual $U(1)$–VOA of rank N. The full automorphism group of $(A(\mathfrak{u}),\Omega_U)$ is $GL(\mathfrak{a})$.*

4.2 Twisted construction

In practice we may wish to take advantage of the fact that the larger group $Spin(\mathfrak{u})$ acts non-trivially on $A(\mathfrak{u})_\theta$. Also, we may wish to be able to realize groups no larger than $SL(\mathfrak{a})$ as symmetry groups of enhanced VOAs. For these goals it is useful to consider the following twisted analogue of the Clifford module construction of VOAs.

Let $\mathfrak{a}$ and $\mathfrak{u}$ be as in the previous section, but restrict now to the case that N is a positive integer divisible by four. Recall from §3.4 that $A(\mathfrak{u})$ is a VOA with superspace decomposition given by

$$A(\mathfrak{u})=A(\mathfrak{u})^0\oplus A(\mathfrak{u})^1, \tag{4.2.1}$$

coinciding with the decomposition into eigenspaces for the action of the parity involution θ. Recall also that the θ-twisted module $A(\mathfrak{u})_\theta$ may be realized as $A(\mathfrak{u})_{\theta,X}$ where X is as in §3.5, and the parity involution acts naturally also on $A(\mathfrak{u})_{\theta,X}$ with eigenspaces $A(\mathfrak{u})^0_{\theta,X}$ and $A(\mathfrak{u})^1_{\theta,X}$. We define the space $\tilde{A}(\mathfrak{u})$ by setting

$$\tilde{A}(\mathfrak{u})=A(\mathfrak{u})^0\oplus A(\mathfrak{u})^0_{\theta,X}. \tag{4.2.2}$$

and we claim that $\tilde{A}(\mathfrak{u})$ admits a structure of self-dual VOA of rank N. In the case that N is divisible by 8, this object is even a VOA, but for now we are more interested in the super case, so we will henceforth assume that N is congruent to 4 modulo 8.

The superspace decomposition of $\tilde{A}(\mathfrak{u})$ coincides with the decomposition in (4.2.2) so that the even subVOA is $A(\mathfrak{u})^0$. We require to exhibit the vertex operators on $\tilde{A}(\mathfrak{u})$, and this may be done as follows. The required vertex operator correspondence $Y:\tilde{A}(\mathfrak{u})\otimes\tilde{A}(\mathfrak{u})\to\tilde{A}(\mathfrak{u})((z))$ is defined already on $\tilde{A}(\mathfrak{u})_{\bar{0}}\otimes\tilde{A}(\mathfrak{u})_{\bar{0}}$ and on $\tilde{A}(\mathfrak{u})_{\bar{0}}\otimes\tilde{A}(\mathfrak{u})_{\bar{1}}$ curtesy of §3.4. For $u\otimes v\in\tilde{A}(\mathfrak{u})_{\bar{1}}\otimes\tilde{A}(\mathfrak{u})_{\bar{0}}$ we define $Y(u,z)v$ by setting

$$Y(u,z)v=e^{zL(-1)}Y(v,-z)u \tag{4.2.3}$$

Suppose now that $u\otimes v\in\tilde{A}(\mathfrak{u})_{\bar{1}}\otimes\tilde{A}(\mathfrak{u})_{\bar{1}}$. Then we define $Y(u,z)v$ by requiring that for any $w\in\tilde{A}(\mathfrak{u})_{\bar{0}}$ we should have

$$\langle Y(u,z)v\mid w\rangle=(-1)^n\langle e^{z^{-1}L(1)}v\mid Y(w,-z^{-1})e^{zL(1)}z^{-2L(0)}u\rangle \tag{4.2.4}$$

whenever $u \in \tilde{A}(\mathfrak{u})_{n-1/2}$ for some $n \in \mathbb{Z}$.

The proof of the following proposition is almost identical to that of Proposition 4.1 in [Dun07] (see also [Hua96]).

Proposition 4.7. *The map* $Y : \tilde{A}(\mathfrak{u}) \otimes \tilde{A}(\mathfrak{u}) \to \tilde{A}(\mathfrak{u})((z))$ *defines a structure of self-dual VOA of rank* N *on* $\tilde{A}(\mathfrak{u})$.

Recall that the even part of $\tilde{A}(\mathfrak{u})$ is $A(\mathfrak{u})^0 = A(\mathfrak{u})_{\bar{0}}$ (so long as $N/4$ is odd).

Proposition 4.8. *For* $N > 4$ *the automorphism group of the VOA* $\tilde{A}(\mathfrak{u})$ *is* $Spin(\mathfrak{u})/\langle \mathfrak{z} \rangle$.

Proof. The proof is very similar to that of Proposition 4.4. We set $G = \mathrm{Aut}(\tilde{A}(\mathfrak{u}))$ and $G_0 = \mathrm{Aut}(\tilde{A}(\mathfrak{u})_{\bar{0}})$ (so that G_0 is the same as in Proposition 4.4) and we let $\phi : G \to G_0$ be the natural map. S is the subgroup of G generated by exponentials $\exp(x_{(0)})$ for $x \in \tilde{A}(\mathfrak{u})_1$, and we find as before that S contains the kernel of ϕ, and $G_0 = \phi(S) \cup \bar{x}\phi(S)$. We claim that the map $\phi : G \to G_0$ is in this case not surjective. We have seen already that $G_0 = O(\mathfrak{u})/\langle \pm\mathrm{Id} \rangle$. If ϕ were surjective then G would be a double cover of $O(\mathfrak{u})/\langle \pm\mathrm{Id} \rangle$ containing S, and the group S is $Spin(\mathfrak{u})/\langle \mathfrak{z} \rangle$ in this case. The only possibility is that $G = Pin(\mathfrak{u})/\langle \mathfrak{z} \rangle$, and this group has no irreducible representation of dimension $2^{N-1} = \dim \tilde{A}(\mathfrak{u})_{N/8}$. We conclude that $\phi(G) = \phi(S)$, so that $\mathrm{Aut}(\tilde{A}(\mathfrak{u})) = S$ is the group $Spin(\mathfrak{u})/\langle \mathfrak{z} \rangle$. □

Note that $\jmath$ belongs to $\tilde{A}(\mathfrak{u})$. It is straightforward to check, as in the proof of Proposition 4.3, that $(\tilde{A}(\mathfrak{u}), \Omega_U)$ is another example of a $U(1)$–VOA.

Proposition 4.9. *Let* N *be a positive integer congruent to* 4 *modulo* 8. *Then the quadruple* $(\tilde{A}(\mathfrak{u}), Y, \mathbf{1}, \Omega_U)$ *is a self-dual* $U(1)$*–VOA of rank* N.

4.3 Special linear groups

Suppose we wish to restrict the symmetry of $\tilde{A}(\mathfrak{u})$ further so as to obtain an action by a group no bigger than the preimage of $SL(\mathfrak{a})$. We can achieve this by including elements such as $\nu_\circ = \mathbf{1}_X$ and $\nu_\bullet = e_\Delta \mathbf{1}_X$ in some new enhanced conformal structure on $\tilde{A}(\mathfrak{u})$.

Proposition 4.10. *The subgroup of* $\mathrm{Aut}(\tilde{A}(\mathfrak{u}), \omega)$ *fixing* $\jmath$, $\nu_\circ$, *and* $\nu_\bullet$ *is the group* $SL(\mathfrak{a})/\langle \pm\mathrm{Id} \rangle$.

Proof. Let H be the group of the statement of the proposition, and recall from Proposition 4.8 that $\mathrm{Aut}(\tilde{A}(\mathfrak{u})) = Spin(\mathfrak{u})/\langle \mathfrak{z} \rangle$. Let $G = \mathrm{Aut}(\tilde{A}(\mathfrak{u}), \Omega_U)$ and $G_0 = \mathrm{Aut}(\tilde{A}(\mathfrak{u})_{\bar{0}}, \Omega_U)$. Then the natural map $\phi : G \to G_0$ is surjective, since by the remark following the proof of Proposition 4.5 every automorphism of $(A(\mathfrak{u})_{\bar{0}}, \Omega_U)$ is inner (and $A(\mathfrak{u})_{\bar{0}} = \tilde{A}(\mathfrak{u})_{\bar{0}}$). In fact, G is the quotient by $\langle \mathfrak{z} \rangle$ of the group generated by automorphisms $\pm\exp(tX_{ij})$ where $t \in \mathbb{C}$ and X_{ij} is the residue of $Y(x_{ij}, z)$ for $x_{ij} = a_i(-\frac{1}{2})a_j^*(-\frac{1}{2})\mathbf{1}$. The exponentials $\exp(tX_{ij})$ generate a copy of $GL(\mathfrak{a})/\langle \pm\mathrm{Id} \rangle$ in $Spin(\mathfrak{u})/\langle \mathfrak{z} \rangle$. Allowing $\pm\exp(tX_{ij})$ we obtain a double cover of this group. Now consider the action of G on $\nu_\circ, \nu_\bullet \in \tilde{A}(\mathfrak{u})_{N/8}$. The subgroup of G fixing $\mathbf{1}_X$ is just the group $GL(\mathfrak{a})/\langle \pm\mathrm{Id} \rangle$. This latter group then preserves all the subspaces $\bigwedge^{2k}(\mathfrak{a})\mathbf{1}_X$ of $\tilde{A}(\mathfrak{u})_{N/8}$, and the action is given by

$$\bar{g} : a_{i_1} \cdots a_{i_{2k}} \mathbf{1}_X \mapsto g(a_{i_1}) \cdots g(a_{i_{2k}}) \mathbf{1}_X \tag{4.3.1}$$

for g a preimage in $GL(\mathfrak{a})$ of $\bar{g} \in GL(\mathfrak{a})/\langle \pm\mathrm{Id} \rangle$. Evidently, the coefficient of $a_\Delta \mathbf{1}_X$ in $\bar{g} a_\Delta \mathbf{1}_X$ is the determinant of either preimage of $\bar{g}$ in $GL(\mathfrak{a})$. The claim follows. □

To understand the local algebras (see §2) arising from the inclusion of vectors like $\nu_\circ$ and $\nu_\bullet$ in an enhanced conformal structure on $\tilde{A}(\mathfrak{u})$ we should calculate the OPEs involving the corresponding vertex operators. This calculation is the content of the following proposition. Let $\nu_+ = \nu_\circ + \nu_\bullet$ and $\nu_- = \mathbf{i}(\nu_\circ - \nu_\bullet)$. Then we have

Proposition 4.11. *The operators $V_+(z) = Y(\nu_+, z)$ and $V_-(z) = Y(\nu_-, z)$ satisfy the following OPEs*

$$V_\pm(z)V_\pm(w) = \pm(-1)^{N/8+1/2} \sum_{k=0}^{N/4-1} \frac{P_{k,-\mathbf{i}J}(w) + P_{k,\mathbf{i}J}(w)}{(z-w)^{N/4-k}} + \text{reg.} \tag{4.3.2}$$

$$V_+(z)V_-(w) = \mathbf{i}(-1)^{N/8+1/2} \sum_{k=0}^{N/4-1} \frac{P_{k,-\mathbf{i}J}(w) - P_{k,\mathbf{i}J}(w)}{(z-w)^{N/4-k}} + \text{reg.} \tag{4.3.3}$$

where $P_{k,\pm\mathbf{i}J}(z)$ is defined to be the coefficient of X^k in

$$:\exp\left(\sum_{m>0} \pm\mathbf{i}J^{(m)}(z)X^m\right): \tag{4.3.4}$$

and $J^{(m)}(z) = \frac{1}{m}D_z^{(m-1)}J(z)$ for $m \in \mathbb{Z}_{>0}$.

We see from Proposition 4.11 that (at least when $N > 4$) if $\Omega \subset \tilde{A}(\mathfrak{u})$ contains ν_+ and ν_- then Ω and $\Omega \cup \{\jmath\}$ determine the same enhanced conformal structure on $\tilde{A}(\mathfrak{u})$. More precisely, the vertex Lie subalgebra of $\operatorname{Sing}\tilde{A}(\mathfrak{u})$ generated by Ω coincides with that generated by $\Omega \cup \{\jmath\}$ (see §2).

We record some of the observations of this section in the following

Theorem 4.12. *For N congruent to 4 modulo 8 and greater than 4, and for $\Omega = \{\omega, \nu_+, \nu_-\}$, the quadruple $(\tilde{A}(\mathfrak{u}), Y, \mathbf{1}, \Omega)$ is a self-dual enhanced $U(1)$–VOA of rank N. The full automorphism group of $(\tilde{A}(\mathfrak{u}), \Omega)$ is the group $SL(\mathfrak{a})/\langle \pm \mathrm{Id} \rangle$.*

5 The Rudvalis group

In this section we realize the sporadic simple group of Rudvalis as symmetries of an enhanced VOA A_{Ru}. In §5.4 we show that the full automorphism group of A_{Ru} is the direct product of a cyclic group of order seven with the Rudvalis group.

Our plan for realizing the Rudvalis group as symmetry of an enhanced VOA is to consider first the enhanced VOA for $SL_N(\mathbb{C})$ for suitable N, constructed in §4, and then to find a single extra (super)conformal generator with which to refine the conformal structure, just to the point that the Rudvalis group becomes visible.

We give two constructions of this conformal generator. The first construction arises directly from the geometry of the Conway–Wales lattice [Con77], and is given in §5.1. The second construction, which is more convenient for computations, is a description in terms of a particular maximal

subgroup of a double cover of the Rudvalis group, and is given in §5.2. Sections 5.1 and 5.2 are independent, and the reader my safely skip one in favor of the other. The approach of §5.1 is certainly more conceptual, and more brief.

The enhanced VOA structure for A_{Ru} is described in §5.3. In §5.4 we determine its symmetry group, and conjecture a characterization.

5.1 Geometric description

Let Λ denote the Conway–Wales lattice [Con77], viewed as a module of rank 28 over the Gaussian integers $\mathbb{Z}[\mathbf{i}]$, and equipped with a non-degenerate Hermitian form denoted $(\cdot\,,\cdot)$. (An explicit description of this lattice is given in the sequel [Dun06a].) We write $\widetilde{R}$ for $\mathrm{Aut}(\Lambda)$, the subgroup of the unitary group of the space $\mathfrak{r} = \mathbb{C} \otimes_{\mathbb{Z}[\mathbf{i}]} \Lambda$ that preserves the set of vectors in Λ. Then $\widetilde{R}$ is a four-fold cover of the Rudvalis group Ru, and may be written as a central product $4 \circ (2.Ru)$ where the 4 is generated by multiplication by $\mathbf{i}$, and the non-trivial central element of the perfect cover $2.Ru$ is multiplication by -1 [Con77]. Let Λ_2 denote the set of type 2 vectors in Λ; i.e. the vectors $\lambda \in \Lambda$ with $(\lambda, \lambda) = 4$. These are also called the *sacred vectors* [Con77]. Counting projectively, there are 4060 sacred vectors in Λ, and $u\lambda$ is a sacred vector whenever λ is sacred and u is a unit in $\mathbb{Z}[\mathbf{i}]$.

For $X \subset \Lambda_2$ we define $\Gamma(X)$ to be the (undirected) graph with vertices $\{v_\lambda\}_{\lambda\in X}$ indexed by the λ in X, and edges such that v_λ is joined to $v_{\lambda'}$ just when $(\lambda, \lambda') = 1$. We define $\Delta(X)$ to be the directed graph with vertices $\{v_\lambda\}_{\lambda\in X}$ and a directed edge from v_λ to $v_{\lambda'}$ just when $(\lambda, \lambda') = \mathbf{i}$.

Let $B \subset \Lambda_2$ be an orbit of size 13 for some element of order 13 in $\widetilde{R}$ such that $\Delta(B)$ has 13 (directed) edges. (All elements of order 13 are conjugate in $\widetilde{R}$ [CCN+85], so it matters not which one we choose.) Then $\Delta(B)$ is in fact an oriented triangulation of the circle, and for each λ in B there is a unique $\lambda' \in B$ such that $(\lambda, \lambda') = \mathbf{i}$. In addition to this, the graph $\Gamma(B)$ has 26 (undirected) edges, and is a triangulation of a Möbius band, or at least, the graph one would obtain by taking a triangulated Möbius band (with 13 triangles) and placing a (graph) edge along the "edge" of each 2-cell, and vertices at the intersection of each (graph) edge.

Definition. We call $B \subset \Lambda_2$ an *M-set* in case $\Gamma(B)$ is a triangulation of a Möbius band (in the above sense), and $\Delta(B)$ is an oriented triangulation of a circle, each triangulation having 13 triangles.

It turns out that any M-set in Λ_2 is an orbit for some element of order 13 in $\widetilde{R}$. Considering the M-sets that are orbits for a fixed element of order 13 in $\widetilde{R}$, we can easily find elements of $\widetilde{R}$ sending one to another. Recalling that there is a single conjugacy class of elements of order 13 in $\widetilde{R}$ we then have

Proposition 5.1. *The M-sets in Λ_2 constitute a single orbit under $\widetilde{R}$.*

For B an M-set, we say that a labeling $B = \{\beta^1, \ldots, \beta^{13}\}$ of the elements of B is *oriented* if we have $(\beta^i, \beta^{i+1}) = \mathbf{i}$ for each $i \in \{1, \ldots, 12\}$. Then all oriented labelings of B are equivalent under cyclic permutations of the β^i, and the expression $\beta^1 \wedge \beta^2 \wedge \cdots \wedge \beta^{13}$ is a well defined element of $\bigwedge^{13}(\mathfrak{r})$ that is independent of the choice of oriented labeling. In Table 1 we give an example of an M-set; we denote this particular example B_*. Our notation for vectors in Λ_2 is similar to that used in [Wil84]: we view these vectors as 7-tuples of complex quaternions as in [Con77], and we present

such an object as a 4×7 array, but with the coefficients of 1, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{l}$ (rather than $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{l}$) appearing in the respective columns. We obtain a particular oriented labeling for B_* by reading the vectors in Table 1 from left to right within rows, from the top row to the bottom. We denote this particular labeling by $B_* = \{\beta_*^1, \dots, \beta_*^{13}\}$.

It turns out that an M-set $B = \{\beta^1, \dots, \beta^{13}\}$ wields much control over the other sacred vectors in Λ, in the sense that, generically, a sacred vector $\lambda \in \Lambda_2$ is determined uniquely among all sacred vectors by the sequence of values (β^k, λ) for $k = 1, \dots, 13$. Important exceptions to this principle are the vectors $\lambda \in \Lambda_2$ satisfying $(B, \lambda) = \{u_0\}$ for some fixed unit u_0 in $\mathbb{Z}[\mathbf{i}]$. (We write (B, λ) as a shorthand for the set $\{(\beta, \lambda) \mid \beta \in B\}$.) For example, for any M-set B there are exactly two vectors μ, μ' in Λ_2 such that $(B, \mu) = (B, \mu') = \{1\}$. Nonetheless, we can tell them apart, since the inner product (μ, μ') is always non-zero and non-real, and hence depends on the order. We call μ and μ' the *neighbors* to B if $(B, \mu) = (B, \mu') = \{1\}$ (and $\mu \neq \mu'$), and we say that μ is the *first neighbor* if $(\mu, \mu') = \mathbf{i}$. The vector μ' is then called the *second neighbor* to B. The first and second neighbors to B_* will be denoted μ_* and μ'_*, respectively. They appear in Table 2.

Consider then the case that $u_0 = 0$. Counting projectively, there are just two vectors satisfying $(B, \lambda) = \{0\}$. Again we can distinguish them, at least up to multiplication by units, for if $\lambda \in \Lambda_2$ satisfies $(B, \lambda) = \{0\}$ then for μ, μ' the neighbors to B, we have either $(\lambda, \mu) = (\lambda, \mu')$ or $(\lambda, \mu) = -(\lambda, \mu')$. We say that λ is a *complement* to B if $(B, \lambda) = \{0\}$, and we say that a complement λ is *positive* (resp. *negative*) if $\frac{(\lambda,\mu)}{(\lambda,\mu')}$ is $+1$ (resp. -1). If λ and λ' are positive and negative complements to an M-set B, then they are not orthogonal. In particular, for a given positive complement λ, there is a unique negative complement λ' such that $(\lambda, \lambda') = 1$. In Table 3 we give a positive complement λ_* for B_*, and a negative complement λ'_* satisfying $(\lambda_*, \lambda'_*) = 1$.

If B is an M-set and λ and λ' are positive and negative complements to B respectively, then there are M-sets for which λ is a negative complement, and λ' is a positive complement. We can single out one of these C say, by defining $C = \{\gamma^1, \dots, \gamma^{13}\}$ to be the set of vectors such that the inner products (β^i, γ^j) take on certain specified values. If B is an M-set, then there is a unique vector γ such that the sequence (β^i, γ) coincides with that given in (5.1.1), and similarly for each cyclic permutation of (5.1.1).

$$(-\mathbf{i}, -\mathbf{i}, 0, 0, \mathbf{i}, -\mathbf{i}, 0, 0, 0, 0, -\mathbf{i}, \mathbf{i}, 0) \tag{5.1.1}$$

We say that $C = \{\gamma^j\}$ is a *right partner* to an M-set B if the γ^j are all the vectors such that the sequence (β^i, γ^j) (with varying i) is a cyclic permutation of (5.1.1). It turns out that C is again an M-set, with the same complements as B, and that a positive complement to C is a negative complement to B, and vice-versa. To prove these statements it suffices to check one example, so we exhibit a right partner to B_* in Table 4. This right partner to B_* will be denoted C_*. We write $\{\gamma_*^1, \dots, \gamma_*^{13}\}$ for the particular oriented labeling of C_* obtained by reading from left to right, and top to bottom in Table 4. If C is a right partner to B, then $-B$ is a right partner to C.

Let us define an undirected graph Ξ whose vertices are pairs $[\lambda, B]$ where B is an M-set and λ is a positive complement to B, and whose edges are determined as follows: the vertices $[\lambda_0, B_0]$ and $[\lambda_1, B_1]$ are joined if any one of the following situations hold.

(Ξ1) $\lambda_0 = u\lambda_1$ and $B_0 = \bar{u}B_1$ for some unit u in $\mathbb{Z}[\mathbf{i}]$.

Table 1: The M-set B_*

$$\left|\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right|$$

$$\frac{1}{2}\left|\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\mathbf{i}} & \bar{1} & \mathbf{i} & \bar{1} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & \bar{\mathbf{i}} & \mathbf{i} \\ 1 & \mathbf{i} & \mathbf{i} & \bar{1} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} \bar{1} & \mathbf{i} & 0 & 0 \\ 0 & 0 & \mathbf{i} & \bar{\mathbf{i}} \\ 0 & \mathbf{i} & 0 & \bar{1} \\ 0 & \mathbf{i} & \bar{\mathbf{i}} & 0 \\ 0 & \bar{1} & 0 & 1 \\ \bar{1} & 1 & 1 & \bar{\mathbf{i}} \\ 0 & \mathbf{i} & \bar{1} & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & 0 & \mathbf{i} & \bar{1} \\ 0 & 0 & \bar{1} & \bar{1} \\ 0 & 1 & 0 & \mathbf{i} \\ \mathbf{i} & 0 & 0 & \mathbf{i} \\ 0 & \mathbf{i} & 0 & \bar{\mathbf{i}} \\ \mathbf{i} & \bar{1} & \mathbf{i} & \mathbf{i} \\ 0 & \mathbf{i} & \bar{1} & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & \mathbf{i} & \mathbf{i} & 0 \\ 0 & \bar{1} & 0 & 1 \\ 1 & 1 & \mathbf{i} & 1 \\ 0 & \mathbf{i} & \bar{1} & 0 \\ 0 & 0 & \bar{\mathbf{i}} & \bar{1} \\ \bar{\mathbf{i}} & \mathbf{i} & 0 & 0 \\ 0 & 1 & 0 & \mathbf{i} \end{matrix}\right|$$

$$\frac{1}{2}\left|\begin{matrix} 1 & 0 & 0 & \bar{1} \\ 0 & 1 & 0 & 1 \\ 1 & \bar{1} & 1 & \bar{\mathbf{i}} \\ 0 & 1 & \mathbf{i} & 0 \\ 1 & \mathbf{i} & 0 & 0 \\ 0 & 0 & 1 & \bar{1} \\ \mathbf{i} & 0 & \bar{1} & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} \bar{1} & 0 & 0 & \mathbf{i} \\ 0 & 0 & 1 & \bar{\mathbf{i}} \\ 0 & 0 & 1 & \bar{1} \\ 0 & \bar{1} & 0 & \mathbf{i} \\ 0 & \mathbf{i} & \bar{\mathbf{i}} & 0 \\ 0 & \mathbf{i} & 0 & \bar{\mathbf{i}} \\ 1 & \bar{1} & \mathbf{i} & \bar{1} \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} \mathbf{i} & \mathbf{i} & \mathbf{i} & \bar{1} \\ 0 & \mathbf{i} & \bar{1} & 0 \\ 0 & 0 & \bar{\mathbf{i}} & \bar{1} \\ 0 & 0 & \bar{\mathbf{i}} & \mathbf{i} \\ 0 & 1 & 0 & \bar{\mathbf{i}} \\ 0 & 1 & 1 & 0 \\ \bar{\mathbf{i}} & 0 & \bar{\mathbf{i}} & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 1 & 0 & 0 & \mathbf{i} \\ 0 & 0 & \bar{1} & \bar{\mathbf{i}} \\ 1 & 1 & 0 & 0 \\ 0 & \bar{1} & 0 & \mathbf{i} \\ 0 & \bar{1} & \bar{1} & 0 \\ 1 & 0 & \bar{1} & 0 \\ \bar{1} & 1 & \bar{1} & \mathbf{i} \end{matrix}\right|$$

$$\frac{1}{2}\left|\begin{matrix} 1 & 0 & \mathbf{i} & 0 \\ 0 & \bar{\mathbf{i}} & \mathbf{i} & 0 \\ 1 & 0 & 1 & 0 \\ \bar{1} & \mathbf{i} & \bar{\mathbf{i}} & \bar{\mathbf{i}} \\ 0 & 1 & \bar{\mathbf{i}} & 0 \\ 0 & 0 & \mathbf{i} & \bar{1} \\ \mathbf{i} & \bar{\mathbf{i}} & 0 & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} \bar{\mathbf{i}} & 0 & 1 & 0 \\ 0 & \bar{\mathbf{i}} & \bar{\mathbf{i}} & 0 \\ 0 & \bar{1} & 0 & \bar{1} \\ 1 & \mathbf{i} & \bar{1} & \bar{1} \\ 0 & \mathbf{i} & \bar{1} & 0 \\ 0 & 0 & 1 & \bar{\mathbf{i}} \\ 0 & 0 & \bar{1} & \bar{1} \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} \bar{1} & \bar{1} & 1 & \mathbf{i} \\ 0 & \mathbf{i} & \bar{1} & 0 \\ 0 & 0 & 1 & \mathbf{i} \\ 0 & 0 & \bar{\mathbf{i}} & \mathbf{i} \\ \mathbf{i} & 0 & 1 & 0 \\ \bar{1} & 0 & 0 & \bar{1} \\ 1 & 0 & \bar{1} & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & \mathbf{i} & \bar{1} & 0 \\ 0 & 0 & \bar{1} & \bar{\mathbf{i}} \\ \mathbf{i} & \bar{\mathbf{i}} & 0 & 0 \\ \mathbf{i} & 0 & \bar{1} & 0 \\ \bar{\mathbf{i}} & 0 & 0 & \bar{\mathbf{i}} \\ 0 & \mathbf{i} & 0 & \mathbf{i} \\ \bar{\mathbf{i}} & \bar{\mathbf{i}} & \bar{1} & \mathbf{i} \end{matrix}\right|$$

Table 2: Neighbors to B_*

$$\mu_* = \frac{1}{2}\begin{vmatrix} 0 & \mathbf{i} & 0 & \bar{1} \\ \mathbf{i} & 0 & 0 & \bar{\mathbf{i}} \\ 1 & 0 & 1 & 0 \\ 1 & \mathbf{i} & \bar{\mathbf{i}} & \mathbf{i} \\ \bar{\mathbf{i}} & 0 & 0 & \bar{1} \\ 0 & 0 & \mathbf{i} & \bar{1} \\ 0 & 0 & \bar{1} & \bar{1} \end{vmatrix} \quad \mu'_* = \frac{1}{2}\begin{vmatrix} 0 & 0 & \mathbf{i} & \mathbf{i} \\ \mathbf{i} & 0 & \bar{1} & 0 \\ 1 & 0 & 0 & \bar{1} \\ \bar{1} & 0 & \bar{1} & 0 \\ \mathbf{i} & \mathbf{i} & \bar{\mathbf{i}} & 1 \\ 0 & \mathbf{i} & 1 & 0 \\ 0 & 0 & \bar{1} & \mathbf{i} \end{vmatrix}$$

Table 3: Complements to B_*

$$\lambda_* = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \lambda'_* = \frac{1}{2}\begin{vmatrix} \bar{1} & 1 & 1 & \bar{\mathbf{i}} \\ 1 & 0 & 0 & \bar{\mathbf{i}} \\ 0 & 0 & \bar{\mathbf{i}} & 1 \\ 1 & \bar{1} & 0 & 0 \\ \bar{\mathbf{i}} & 0 & 1 & 0 \\ 1 & 0 & 0 & \bar{1} \\ \mathbf{i} & 0 & \bar{\mathbf{i}} & 0 \end{vmatrix}$$

Table 4: The M-set C_*

$$\frac{1}{2}\left|\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & \mathbf{i} & 0 & \bar{1} \\ \mathbf{i} & 0 & 0 & \bar{\mathbf{i}} \\ \bar{1} & 0 & \bar{1} & 0 \\ 1 & \mathbf{i} & \bar{\mathbf{i}} & \mathbf{i} \\ \mathbf{i} & 0 & 0 & 1 \\ 0 & 0 & \bar{\mathbf{i}} & 1 \end{matrix}\right|$$

$$\frac{1}{2}\left|\begin{matrix} \bar{1} & \bar{\mathbf{i}} & 0 & 0 \\ 0 & 0 & \bar{1} & 1 \\ \mathbf{i} & 0 & \bar{1} & 0 \\ \bar{1} & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & \bar{1} & 1 & \bar{\mathbf{i}} \\ 0 & 1 & \mathbf{i} & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & \bar{\mathbf{i}} & 0 & \bar{1} \\ 0 & \bar{1} & 1 & 0 \\ 0 & 1 & 0 & \bar{1} \\ \mathbf{i} & \bar{\mathbf{i}} & \bar{1} & \bar{\mathbf{i}} \\ \mathbf{i} & 0 & 0 & 1 \\ 0 & 0 & \bar{\mathbf{i}} & \bar{1} \\ 0 & 0 & \bar{\mathbf{i}} & \mathbf{i} \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & 0 & \mathbf{i} & 1 \\ 0 & 0 & \bar{\mathbf{i}} & \mathbf{i} \\ 0 & \bar{\mathbf{i}} & 0 & \bar{1} \\ 0 & \bar{1} & 1 & 0 \\ 0 & \bar{1} & 0 & 1 \\ \bar{\mathbf{i}} & \mathbf{i} & 1 & \mathbf{i} \\ \mathbf{i} & 0 & 0 & 1 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & 1 & 0 & \mathbf{i} \\ 0 & 1 & \bar{1} & 0 \\ \bar{\mathbf{i}} & 0 & \mathbf{i} & 0 \\ \bar{1} & \bar{1} & \bar{1} & \bar{\mathbf{i}} \\ 0 & \mathbf{i} & 1 & 0 \\ 0 & 0 & \bar{\mathbf{i}} & 1 \\ 0 & 0 & \bar{\mathbf{i}} & \bar{\mathbf{i}} \end{matrix}\right|$$

$$\frac{1}{2}\left|\begin{matrix} \mathbf{i} & \bar{\mathbf{i}} & \mathbf{i} & 1 \\ 0 & \mathbf{i} & \bar{1} & 0 \\ \mathbf{i} & \bar{1} & 0 & 0 \\ 0 & 0 & \mathbf{i} & \bar{\mathbf{i}} \\ \bar{1} & 0 & \bar{\mathbf{i}} & 0 \\ \mathbf{i} & 0 & 0 & \bar{\mathbf{i}} \\ 0 & \mathbf{i} & 0 & \mathbf{i} \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} \bar{1} & \bar{1} & 0 & 0 \\ 0 & \bar{\mathbf{i}} & 0 & 1 \\ 0 & \mathbf{i} & \bar{\mathbf{i}} & 0 \\ 0 & \bar{\mathbf{i}} & 0 & \bar{\mathbf{i}} \\ \mathbf{i} & \mathbf{i} & \bar{\mathbf{i}} & 1 \\ 1 & 0 & 0 & \mathbf{i} \\ \bar{\mathbf{i}} & 1 & 0 & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & 1 & 0 & \bar{\mathbf{i}} \\ 0 & \bar{1} & \bar{1} & 0 \\ 0 & 1 & 0 & \bar{1} \\ \bar{\mathbf{i}} & \bar{\mathbf{i}} & \mathbf{i} & \bar{1} \\ \mathbf{i} & 0 & 0 & 1 \\ \bar{1} & \mathbf{i} & 0 & 0 \\ \mathbf{i} & \bar{\mathbf{i}} & 0 & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & 0 & 1 & \mathbf{i} \\ 0 & 0 & \mathbf{i} & \mathbf{i} \\ 0 & \bar{\mathbf{i}} & 0 & 1 \\ \bar{1} & 0 & 0 & \bar{1} \\ 0 & \bar{1} & 0 & 1 \\ 1 & \mathbf{i} & 1 & 1 \\ 0 & \bar{1} & \bar{\mathbf{i}} & 0 \end{matrix}\right|$$

$$\frac{1}{2}\left|\begin{matrix} 0 & 0 & \bar{1} & 1 \\ 0 & \mathbf{i} & 0 & \bar{1} \\ 0 & \bar{1} & \bar{1} & 0 \\ \bar{1} & 0 & \bar{1} & 0 \\ \mathbf{i} & \mathbf{i} & 1 & \bar{\mathbf{i}} \\ 1 & 0 & 0 & \bar{\mathbf{i}} \\ \mathbf{i} & 1 & 0 & 0 \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & \bar{\mathbf{i}} & \mathbf{i} & 0 \\ 0 & 1 & 0 & 1 \\ \mathbf{i} & \mathbf{i} & \bar{1} & \mathbf{i} \\ 0 & \bar{\mathbf{i}} & \bar{1} & 0 \\ 0 & 0 & \bar{\mathbf{i}} & 1 \\ \bar{\mathbf{i}} & \bar{\mathbf{i}} & 0 & 0 \\ 0 & \bar{1} & 0 & \mathbf{i} \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} 0 & \bar{1} & 0 & \bar{\mathbf{i}} \\ 0 & \bar{\mathbf{i}} & \bar{\mathbf{i}} & 0 \\ \bar{\mathbf{i}} & 0 & \bar{\mathbf{i}} & 0 \\ \bar{1} & \bar{1} & \mathbf{i} & 1 \\ \mathbf{i} & 0 & 0 & 1 \\ \bar{1} & \mathbf{i} & 0 & 0 \\ 0 & 0 & \bar{\mathbf{i}} & \mathbf{i} \end{matrix}\right| \quad \frac{1}{2}\left|\begin{matrix} \bar{\mathbf{i}} & 1 & \mathbf{i} & \mathbf{i} \\ 0 & \bar{\mathbf{i}} & \bar{1} & 0 \\ 0 & 0 & \bar{1} & \bar{\mathbf{i}} \\ 0 & 0 & 1 & \bar{1} \\ \mathbf{i} & 0 & 1 & 0 \\ 0 & \mathbf{i} & \bar{\mathbf{i}} & 0 \\ 0 & \bar{1} & 0 & 1 \end{matrix}\right|$$

(Ξ2) $\lambda_0 = \lambda_1$ and $(\lambda_0, \mu_0) = (\lambda_0, \mu_1)$ where μ_0 is the first neighbor to B_0, and μ_1 is the first neighbor to B_1.

(Ξ3) λ_1 is the negative complement to B_0 for which $(\lambda_0, \lambda_1) = -\mathbf{i}$, and B_1 is the right partner to B_0.

(For u a unit and B an M-set, uB denotes the M-set $\{u\beta \mid \beta \in B\}$.) Given a vertex $[\lambda, B]$ in Ξ, we write $a_{[\lambda,B]}$ for the element of $\bigwedge^{14}(\mathfrak{r})$ given by

$$a_{[\lambda,B]} = \lambda \wedge \beta^1 \wedge \cdots \wedge \beta^{13} \tag{5.1.2}$$

where $\{\beta^1, \ldots, \beta^{13}\}$ is any oriented labeling of B. Note that $a_{[\lambda,B]}$ is independent of the oriented labeling chosen. Observe also that $\widetilde{R}$ acts naturally on the pairs $[\lambda, B]$, and on $\bigwedge^{14}(\mathfrak{r})$, and these actions are compatible in the sense that $ga_{[\lambda,B]} = a_{[g\lambda,gB]}$ for $g \in \widetilde{R}$.

We are now ready to define the extra superconformal generator ϱ, which will determine the enhanced VOA structure that we will utilize in §5.3. Let Ξ_* be the connected component of Ξ containing the vertex $[\lambda_*, B_*]$, and define $\varrho \in \bigwedge^{14}(\mathfrak{r})$ by setting

$$\varrho = \sum_{[\lambda,B]\in\Xi_*} a_{[\lambda,B]} \tag{5.1.3}$$

where $a_{[\lambda,B]}$ is defined in (5.1.2).

We claim that ϱ is invariant for the natural action of the Rudvalis group on $\bigwedge^{14}(\mathfrak{r})$. This claim will follow from Proposition 5.5 below, and in preparation for this we now introduce some notation and a few lemmas. Let R be the unique subgroup of $\widetilde{R} = \mathrm{Aut}(\Lambda)$ such that R is isomorphic to $2.Ru$. (If R' is another such group, then its intersection with R contains all elements of odd order in R. These elements generate a group that is non-central and normal in both R' and R, and hence is both R' and R.) For $\lambda \in \Lambda$, we write $\mathrm{Fix}(\lambda)$ for the subgroup of $\widetilde{R}$ consisting of g for which $g\lambda = \lambda$. Then R contains $\mathrm{Fix}(\lambda)$ for all λ in Λ_2.

Lemma 5.2. *For $\lambda \in \Lambda_2$ the group $\mathrm{Fix}(\lambda)$ acts projectively transitively on M-sets for which λ is a positive complement. That is, if B_1 and B_2 are M-sets for which λ is a positive complement then there exists $g \in \mathrm{Fix}(\lambda)$ and a unit u such that $gB_1 = uB_2$.*

Lemma 5.3. *Let B be an M-set, and let ν_1 and ν_2 be positive and negative complements to B, respectively, with $(\nu_1, \nu_2) = -\mathbf{i}$. Then there exists an element $s \in R$ with $s^2 = -\mathrm{Id}$ such that $s : \nu_1 \mapsto \nu_2$ and $s(B) = C$ where C is the right partner to B.*

Lemma 5.4. *If $\nu_0, \nu_2 \in \Lambda_2$ and ν_2 is not a scalar multiple of ν_0 then there exists a vector $\nu_1 \in \Lambda_2$ such that $(\nu_0, \nu_1) = -\mathbf{i}$ and $(\nu_1, \nu_2) = -\mathbf{i}$.*

Proposition 5.5. *If $g \in R$ and $[\lambda_0, B_0] \in \Xi$ then $[\lambda_0, B_0]$ and $[g\lambda_0, gB_0]$ belong to the same connected component of Ξ.*

Proof. We have several different cases to treat, depending on the value of $(\lambda_0, g\lambda_0)$. For given $[\lambda_0, B_0]$ let us write Ξ_0 for the connected component of Ξ containing $[\lambda_0, B_0]$. If we establish that $[g\lambda_0, gB_0] \in \Xi_0$ whenever $(\lambda_0, g\lambda_0) = a$ say, then the same holds also when $(\lambda_0, g\lambda_0) = -a$, since then $[-g\lambda_0, -gB_0] \in \Xi_0$, and $[-g\lambda_0, -gB_0]$ is joined to $[g\lambda_0, gB_0]$ by $(\Xi 1)$.

Case (i): $(\lambda_0, g\lambda_0) = 4$. This is the case that $g \in \mathrm{Fix}(\lambda_0)$. Let μ_0 be the first neighbor to B_0 and observe that $g\mu_0$ is the first neighbor to gB_0. Then we have $(\lambda_0, \mu_0) = (g\lambda_0, g\mu_0) = (\lambda_0, g\mu_0)$ so that $[\lambda_0, B_0]$ is joined to $[g\lambda_0, gB_0] = [\lambda_0, gB_0]$ curtesy of $(\Xi 2)$.

Case (ii): $(\lambda_0, g\lambda_0) = -\mathbf{i}$. By Lemma 5.2 there exists $g_0 \in \mathrm{Fix}(\lambda_0)$ such that λ_0 and $g\lambda_0$ are positive and negative complements to $B_1 := g_0B_0$, respectively. By Lemma 5.3 there exists $s_1 \in R$ such that $s_1^2 = -\mathrm{Id}$ and $s_1\lambda_0 = g\lambda_0$, and $B_2 := s_1(B_1)$ is the right partner to B_1. Now set $h = s_1g_0$ and $g_2 = gh^{-1}$, and observe that $g_2 \in \mathrm{Fix}(g\lambda_0)$. The decomposition $g = g_2s_1g_0$ corresponds in the following way to a path in Ξ_0. We set $[\lambda_1, B_1] = [g_0\lambda_0, g_0B_0]$, $[\lambda_2, B_2] = [s_1\lambda_1, s_1B_1]$, and $[\lambda_3, B_3] = [g_2\lambda_2, g_2B_2]$. Then the vertices $[\lambda_0, B_0]$ and $[\lambda_1, B_1]$ are joined curtesy of the argument in case (i), the vertices $[\lambda_1, B_1]$ and $[\lambda_2, B_2]$ are joined thanks to $(\Xi 3)$, and the vertices $[\lambda_2, B_2]$ and $[\lambda_3, B_3] = [g\lambda_0, gB_0]$ are joined thanks again to case (i).

The remaining cases are dealt with in an analogous way to case (ii), except that we may require a longer decomposition $g = g_{2k}s_{2k-1}\cdots g_2s_1g_0$, where $k+1$ is the length of a sequence $\lambda_0, \lambda_2, \ldots, \lambda_{2k} \in \Lambda_2$ such that $\lambda_{2k} = g\lambda_0$ and $(\lambda_{2m}, \lambda_{2m+2}) = -\mathbf{i}$ for each m. In each case, the existence of a such a sequence is insured by Lemma 5.4.

Case (iii): $(\lambda_0, g\lambda_0) \in \{0, 1\}$. Set $\lambda_4 = g\lambda_0$. By Lemma 5.4 there exists $\lambda_2 \in \Lambda_2$ such that $(\lambda_0, \lambda_2) = -\mathbf{i}$ and $(\lambda_2, \lambda_4) = -\mathbf{i}$. As indicated above, we proceed just as in case (ii). Pick $g_0 \in \mathrm{Fix}(\lambda_0)$ such that λ_0 and λ_2 are positive and negative complements to $B_1 := g_0B_0$, respectively, and choose $s_1 \in R$ such that $s_1^2 = -\mathrm{Id}$ and $s_1\lambda_0 = \lambda_2$ and $B_2 := s_1(B_1)$ is the right partner to B_1. Repeating this, pick $g_2 \in \mathrm{Fix}(\lambda_2)$ such that λ_2 and λ_4 are positive and negative complements to $B_3 := g_2B_2$, respectively, and choose $s_3 \in R$ such that $s_3^2 = -\mathrm{Id}$ and $s_2\lambda_2 = \lambda_4$ and $B_4 := s_3(B_3)$ is the right partner to B_3. Set $h = s_3g_2s_1g_0$ and observe that $g_4 := gh^{-1}$ fixes $\lambda_4 = g\lambda_0$. Consider the sequence $[\lambda_k, B_k]$ in Ξ obtained by setting $\lambda_{2k+1} = \lambda_{2k}$ for $0 \le k \le 2$, and $B_5 = g_4B_4$, so that

$$
\begin{aligned}
[\lambda_1, B_1] &= [g_0\lambda_0, g_0B_0] \\
[\lambda_2, B_2] &= [s_1\lambda_1, s_1B_1] \\
&\vdots \\
[\lambda_5, B_5] &= [g_4\lambda_4, g_4B_4]
\end{aligned}
\tag{5.1.4}
$$

and $[\lambda_5, B_5] = [g\lambda_0, gB_0]$. Then $[\lambda_m, B_m]$ is joined to $[\lambda_{m+1}, B_{m+1}]$ by an application of case (i) for m even, and by $(\Xi 3)$ for m odd.

Case (iv): $(\lambda_0, g\lambda_0) = 4\mathbf{i}$. In this case we require a sequence $\lambda_0, \lambda_2, \ldots, \lambda_6$ of length 4. Choose $\lambda_4 \in \Lambda_2$ such that $(\lambda_0, \lambda_4) = -1$. By Lemma 5.4 there exists $\lambda_2 \in \Lambda_2$ such that $(\lambda_0, \lambda_2) = (\lambda_2, \lambda_4) = -\mathbf{i}$. Let $\lambda_6 = -\mathbf{i}\lambda_0 = g\lambda_0$. Then by construction, $(\lambda_4, \lambda_6) = -\mathbf{i}$ also. This furnishes a sequence $\lambda_0, \ldots, \lambda_6$ as required. We then proceed as in case (iii) using a decomposition of the form $g = g_6s_5g_4s_3g_2s_1g_0$ and the corresponding length 8 sequence of vertices in Ξ, each one joined to its successor thanks to alternating applications of case (i) and $(\Xi 3)$.

We have accounted for all cases, and the proof is complete. $\square$

Proposition 5.5 shows that the $a_{[\lambda,B]}$ for $[\lambda, B]$ in Ξ_* — which are the summands in the expression for ϱ in (5.1.3) — constitute a union of orbits for R in $\bigwedge^{14}(\mathfrak{r})$. Of course, the central element $-\mathrm{Id}$ in R acts trivially on $\bigwedge^{14}(\mathfrak{r})$, so we may view this set as a union of orbits for a copy $R/\langle\pm\mathrm{Id}\rangle$ of the simple group Ru. We have established

Theorem 5.6. *The vector ϱ is invariant for the action of the Rudvalis group on $\bigwedge^{14}(\mathfrak{r})$.*

In fact the summands of ϱ are just a single orbit for the Rudvalis group, since from the proof of Proposition 5.5 we see that two vertices of Ξ are joined only if they define the same element of $\bigwedge^{14}(\mathfrak{r})$ (this is condition (Ξ1)), or if they are joined by some element of R (these are conditions (Ξ2) and (Ξ3)).

5.2 Monomial description

We now give a description of ϱ in terms of the monomial action of a group which will turn out to be a maximal subgroup of a double cover of the Rudvalis group. This group has the shape $2^7.G_2(2)$, and we will refer to the copy we construct in §5.2.2 as *the monomial group*. The quotient group $G_2(2)$ may be viewed as the symmetry group of a certain algebraic structure arising from the E_8 lattice, and in §5.2.1 we review these relationships. The extra superconformal generator must admit a description in terms of invariants for the monomial group. We discuss these monomial invariants, and conclude the construction in §5.2.3.

5.2.1 Cayley algebra

Let $\Pi = \mathrm{PG}(1,7) = \{\infty, 0, 1, 2, 3, 4, 5, 6\}$ be a copy of the projective line over $\mathbb{F}_7$. The group $L_2(7)$ acts doubly transitively on Π by permutations. Let $\mathfrak{h}$ be a real vector space of dimension 8 with positive definite bilinear form $\langle\cdot\,,\cdot\rangle$, and let $\{h_i\}_{i\in\Pi}$ be an orthonormal basis for $\mathfrak{h}$ indexed by the set Π. The $\mathbb{Z}$-lattice in $\mathfrak{h}$ generated by the vectors of the form $h_i \pm h_j$ is a lattice of type D_8, and adding the vector $\frac{1}{2}\sum_i h_i$ to these generators, we obtain a copy of the E_8 lattice, the unique up to isomorphism self-dual even lattice of rank 8, and we denote it by Λ.

A quadratic form q on Λ is defined by setting $q(\lambda) = \langle\lambda,\lambda\rangle/2$. Since Λ is even, q takes values in $\mathbb{Z}$ and maps 2Λ to $4\mathbb{Z}$. Thus q induces an $\mathbb{Z}/2 = \mathbb{F}_2$ valued quadratic form $\bar{q}$ on the quotient group $\bar{\Lambda} = \Lambda/2\Lambda$.

Let us set $\mathbf{1} = \frac{1}{2}\sum_i h_i \in \Lambda$. Then Λ supports a structure of non-associative $\mathbb{Z}$-algebra for which $\mathbf{1}$ is a unit; this algebra is known as the integral Cayley algebra, and is isomorphic to a maximal integral order in the Octonion algebra. The algebra structure may be defined in the following way [CCN+85]. We impose the relations

$$2h_n^2 = h_n - \mathbf{1}, \tag{5.2.1}$$

$$2h_\infty h_0 = \mathbf{1} - h_3 - h_5 - h_6, \tag{5.2.2}$$

$$2h_0 h_\infty = \mathbf{1} - h_2 - h_1 - h_4, \tag{5.2.3}$$

and the images of these under the action of $L_2(7)$.

The subspace 2Λ is an ideal for this algebra structure, and we thus obtain a Cayley algebra over $\mathbb{F}_2$ on the space $\bar{\Lambda}=\Lambda/2\Lambda$. We write $\lambda\equiv\lambda'$ when $\lambda+2\Lambda=\lambda'+2\Lambda$ in $\bar{\Lambda}$. Note that $2h_i\in\Lambda\setminus 2\Lambda$ so that $2h_i$ is not zero in $\bar{\Lambda}$. On the other hand, $h_i-h_j\in\Lambda$ so that $2h_i\cong 2h_j$ for all $i,j\in\Pi$. The full automorphism group of Λ is W_{E_8}, the Weyl group of E_8. The center of W_{E_8} acts trivially on $\bar{\Lambda}$, and the subgroup of $W_{E_8}/\langle\pm\mathrm{Id}\rangle$ preserving the $\mathbb{F}_2$ Cayley algebra structure on $\bar{\Lambda}$ is isomorphic to the exceptional group of Lie type $G_2(2)$. This group contains the simple group $G_2(2)'\cong U_3(3)$ with index two.

Reducing the bilinear form on Λ modulo $2\mathbb{Z}$ we obtain an $\mathbb{F}_2$ valued alternating bilinear form b on $\bar{\Lambda}$. We then have $b(x,y)=\bar{q}(x+y)+\bar{q}(x)+\bar{q}(y)$ for $x,y\in\bar{\Lambda}$. One can check the following

Proposition 5.7. *We have $xy=yx$ in $\bar{\Lambda}$ just when $b(x,y)=0$.*

We have the following computations in the integral Cayley algebra Λ.

$$\begin{aligned}(h_\infty+h_0)^2&=\tfrac12(h_\infty^2+h_\infty h_0+h_0h_\infty+h_0^2)\\&=\tfrac12(h_\infty-\mathbf{1}+\mathbf{1}-h_3-h_5-h_6+\mathbf{1}-h_2-h_1-h_4+h_0-\mathbf{1})\\&=-\mathbf{1}+(h_\infty+h_0)\end{aligned}\tag{5.2.4}$$

$$\begin{aligned}(h_\infty-h_0)^2&=\tfrac12(h_\infty^2-h_\infty h_0-h_0h_\infty+h_0^2)\\&=\tfrac12(h_\infty-\mathbf{1}-\mathbf{1}+h_3+h_5+h_6-\mathbf{1}+h_2+h_1+h_4+h_0-\mathbf{1})\\&=-\mathbf{1}\end{aligned}\tag{5.2.5}$$

The first computation (5.2.4) shows that (h_i+h_j) has order three in $\bar{\Lambda}$ and inverse $\mathbf{1}-(h_i+h_j)$ for all $i\neq j\in\Pi$. We obtain 28 pairs $\{a,a^{-1}\}$ of order three elements in $\bar{\Lambda}$ in this way. When a is a cube root of unity we refer to a pair of the form $\{a,a^{-1}\}$ as a *cube root pair*, and we write Δ for the set of all cube root pairs in the $\mathbb{F}_2$ Cayley algebra $\bar{\Lambda}$. The map $x\mapsto axa^{-1}$ is an algebra automorphism of $\bar{\Lambda}$ whenever $a\in\bar{\Lambda}$ has order three.

From the second computation (5.2.5) we see that $(h_i-h_j)^2$ is an involution in $\bar{\Lambda}$, and $(\mathbf{1}+h_i-h_j)^2\equiv 0$ for all $i\neq j\in\Pi$. There are 28 involutions of this form, but we obtain another 35 involutions like $\mathbf{1}-(h_i+h_j+h_k+h_l)$. Thus we have $28+35=63$ pairs $\{x,y\}\neq\{0,\mathbf{1}\}$ in $\bar{\Lambda}$ such that $x+y\equiv\mathbf{1}$ and one element of the pair is an involution. Note that if x is an involution in $\bar{\Lambda}$ and $y\equiv\mathbf{1}+x$ then $y^2\equiv 0$. In each case the involution in the pair is distinguished by having non-trivial norm and by being orthogonal to $\mathbf{1}$.

The group $G_2(2)$ acts transitively on the 63 involutions, and also on the 28 cube root pairs Δ.

If x,y are involutions in $\bar{\Lambda}$ then $x+y+\mathbf{1}$ is also an involution just when $b(x,y)=0$. In this case we have $x+y+z=\mathbf{1}=x^2=y^2=z^2$ for $z=x+y+\mathbf{1}$, and also that b vanishes on $\{\mathbf{1},x,y,z\}$. We refer to such a triple $\{x,y,z\}$ as an *isotropic line* in $\bar{\Lambda}$. A minimal representative in Λ for an involution in $\bar{\Lambda}$ either takes the form $(1\bar{1}0^6)$ or $\frac12(1^4\bar{1}^4)$, so that any involution is orthogonal to precisely 30 other involutions, and there are thus $63{\cdot}30/6=315$ isotropic lines in $\bar{\Lambda}$.

Suppose that $\{x,y,z\}$ is an isotropic line in $\bar{\Lambda}$ and that $xy=z$. Then $\{\mathbf{1},x,y,z\}$ is a multiplicative four group in $\bar{\Lambda}$. Indeed, commutativity follows from isotropy by Proposition 5.7, and from $xy=z$ we deduce

$$yz=y(x+y+\mathbf{1})=z+\mathbf{1}+y=x,\tag{5.2.6}$$

$$zx=z(y+z+\mathbf{1})=x+\mathbf{1}+z=y.\tag{5.2.7}$$

In this case we say that $\{x, y, z\}$ is an *isotropic ring* in $\bar{\Lambda}$. Each involution belongs to just 3 isotropic four groups in $\bar{\Lambda}$, so just $63{\cdot}3/3 = 63$ of the isotropic lines are isotropic rings.

For any given involution x say there are exactly 24 cube roots of unity that are not orthogonal to x. We have $b(x, a) = b(x, a^{-1})$ so that these 24 cube roots constitute 12 cube root pairs. We call the set of cube root pairs incident to an involution x the *dozen* associated to x. Let $l = \{x, y, z\}$ be an isotropic line. Then the intersection of the dozens associated to the involutions in l is a set of four inverse pairs of cube roots of unity. We refer to the set of inverse pairs incident to an isotropic line l as the *quartet* associated to l. In the case that l is a ring, we say that the quartet associated to l is a *ringed quartet*.

We refer to two distinct inverse pairs as a *couple*. Then any couple belongs to five quartets, and thus determines five isotropic lines. These five lines intersect in a single involution which in turn determines a dozen. Thus we find that each couple determines a unique dozen. On the other hand, each couple belongs to just one ringed quartet. Six different couples determine the same dozen, just as $\binom{4}{2} = 6$ different couples determine the same ringed quartet. The six couples corresponding to a given dozen constitute a partition of the 12 inverse pairs of that dozen into disjoint couples.

Remark. The combinatorial concepts that arise here from the action of $G_2(2)$ on the $\mathbb{F}_2$ Cayley algebra have direct analogues in the Conway-Wales lattice [Con77]; a certain self-dual lattice of rank 56 whose automorphism group is a quadruple cover of the Rudvalis group. We intentionally name these concepts so that there is resonance with the nomenclature of [Con77].

5.2.2 Monomial group

In order to specify subsets of Δ and so forth in a convenient way, we follow [Wil84] and arrange the elements of Δ in to a 3×3 grid where each block has three elements, and a distinguished element ∞ is placed just below the center bottom block of the 3×3 grid. We suppose that Δ has been enumerated $\Delta = \{1, \ldots, 27, \infty\}$ and that the set Δ is arranged in the following way, on what we from now on refer to as the Δ-grid.

$$\begin{array}{|c|c|c|}\hline \begin{matrix} & 1 & \\ 2 & & 3\end{matrix} & \begin{matrix} & 4 & \\ 5 & & 6\end{matrix} & \begin{matrix} & 7 & \\ 8 & & 9\end{matrix} \\ \hline \begin{matrix} & 10 & \\ 11 & & 12\end{matrix} & \begin{matrix} & 13 & \\ 14 & & 15\end{matrix} & \begin{matrix} & 16 & \\ 17 & & 18\end{matrix} \\ \hline \begin{matrix} & 19 & \\ 20 & & 21\end{matrix} & \begin{matrix} & 22 & \\ 23 & & 24\end{matrix} & \begin{matrix} & 25 & \\ 26 & & 27\end{matrix} \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{\infty} & \multicolumn{1}{c}{} \end{array} \tag{5.2.8}$$

Now we may specify a subset of Δ by suitably placing asterisks within a copy of the Δ-grid. Each block has a middle, left, and right element, and we sometimes write M, L, or R, respectively, within a block as a short hand for

$$\begin{matrix} & * & \\ \cdot & & \cdot \end{matrix}\,, \quad \begin{matrix} & \cdot & \\ * & & \cdot \end{matrix}\,, \quad \text{and} \quad \begin{matrix} & \cdot & \\ \cdot & & * \end{matrix}\,, \tag{5.2.9}$$

respectively. The notation LM, LR, MR, and LMR, within a block is to be interpreted in a similar way. The symbol $\cdot$ will denote either an empty position in the Δ-grid, or an empty block.

The arrangement of elements of Δ within the Δ-grid may be effected in such a way that the three elements within any block are exactly the triples of inverse pairs that complete ∞ to a ringed quartet. Choosing one of the blocks, the center block for example, there are exactly three dozens that contain the ringed quartet consisting of the pair ∞ and the pairs of that block. These three dozens give us a partition of the 24 elements outside ∞ and the chosen block into three disjoint sets of size eight. These eight sets necessarily each have one element from each block outside the chosen one, and we may assume that the coordinates of any one of these three dozens are all labeled by the same letter, M, L or R, so that the three dozens in question may be written as follows.

$$\begin{array}{c}\begin{array}{|c|c|c|}\hline M & M & M \\ \hline M & LMR & M \\ \hline M & M & M \\ \hline\end{array} \\ * \end{array}\quad\begin{array}{c}\begin{array}{|c|c|c|}\hline L & L & L \\ \hline L & LMR & L \\ \hline L & L & L \\ \hline\end{array} \\ * \end{array}\quad\begin{array}{c}\begin{array}{|c|c|c|}\hline R & R & R \\ \hline R & LMR & R \\ \hline R & R & R \\ \hline\end{array} \\ * \end{array} \tag{5.2.10}$$

Let $\mathfrak{r}$ be a complex vector space of dimension 28 with positive definite Hermitian form $(\cdot\,,\cdot)$ and an orthonormal basis $\{a_i\}_{i\in\Delta}$ indexed by Δ. Let $\mathfrak{r}^*$ be the dual space to $\mathfrak{r}$ with dual basis $\{a_i^*\}_{i\in\Delta}$, and set $\mathfrak{s}=\mathfrak{r}\oplus\mathfrak{r}^*$. We extend the Hermitian form on $\mathfrak{r}$ in the natural way to $\mathfrak{s}$. Similar to §3.5, we equip $\mathfrak{s}$ with the symmetric bilinear form denoted $\langle\cdot\,,\cdot\rangle$ which is just the natural form arising from the pairing between $\mathfrak{r}$ and $\mathfrak{r}^*$ scaled by a factor of $1/2$. We also suppose that $\mathcal{E}=\{e_i,e_{i'}\}_{i\in\Delta}$ is the orthonormal basis, with respect to the bilinear form $\langle\cdot\,,\cdot\rangle$, on $\mathfrak{s}$ determined as in §3.5, and that X is the subgroup of $Spin(\mathfrak{s})$ generated by the expressions $\mathbf{i}e_ie_{i'}$. Later in §5.3 we will define the superspace underlying A_{Ru} by setting

$$A_{Ru}=A(\mathfrak{s})^0\oplus A(\mathfrak{s})^0_{\theta,X} \tag{5.2.11}$$

so that the VOA underlying A_{Ru} coincides with that constructed for the group $SL_{28}(\mathbb{C})/\langle\pm\mathrm{Id}\rangle$ in §4.

We now commence the construction of the monomial group; a subgroup of $SU(\mathfrak{r})$ of the shape $2^7.G_2(2)$.

Let A denote the group of sign changes on coordinates corresponding to dozens in Δ and their complements. That is, for each dozen $D\subset\Delta$ there is an element $\epsilon_D\in A$ such that ϵ_D acts as -1 on the a_i such that $i\in D$, and fixes the other basis vectors. Generators for the dozens (the dozens and their complements constitute a doubly even code $\mathcal{D}$ on Δ) are given in Table 5. These involutions together with the symmetry which is -1 on all coordinates generate a group A of the shape 2^7.

Let P denote the group consisting of just those coordinate permutations corresponding to the elements of $G_2(2)$ that fix the cube root pair labeled ∞. Then P has the shape $3^{1+2}.Q_8.2$, and A and P together generate a split extension $2^7:3^{1+2}.Q_8.2$. Conjugation by an element a of the $\mathbb{F}_2$ Cayley algebra $\bar{\Lambda}$ is an automorphism of $\bar{\Lambda}$ when a is a cube root of unity. Let a_∞ be one of the cube roots in the pair labeled ∞. Then conjugation by a_∞ fixes ∞ and stabilizes each block, and we may assume that it permutes the points within each block as (MLR). In keeping with the notation of [Con77] and [Wil84] we denote this permutation by Q. Remaining generators are given

Table 5: Dozens generating $\mathcal{D}$

M	M	M
M	LMR	M
M	M	M

$*$

L	L	L
L	LMR	L
L	L	L

$*$

MR	$\cdot$	LM
LR	$\cdot$	LR
LM	$\cdot$	MR

$\cdot$

LM	$\cdot$	LR
MR	$\cdot$	MR
LR	$\cdot$	LM

$\cdot$

LR	LM	$\cdot$
LR	LR	$\cdot$
LR	MR	$\cdot$

$\cdot$

MR	LR	$\cdot$
MR	MR	$\cdot$
MR	LM	$\cdot$

$\cdot$

by the permutations

$$\begin{aligned} N_0 = (1,19,25,7)(2,20,26,8)(3,21,27,9) \\ (4,10,22,16)(5,11,23,17)(6,12,24,18), \end{aligned} \tag{5.2.12}$$

$$\begin{aligned} N_{356} = (1,22,25,4)(2,23,26,5)(3,24,27,6) \\ (7,16,19,10)(8,17,20,11)(9,18,21,12), \end{aligned} \tag{5.2.13}$$

$$\begin{aligned} F_{03} = (1,19)(4,22)(7,25)(2,21)(3,20)(5,24) \\ (6,23)(8,27)(9,26)(11,12)(14,15)(17,18), \end{aligned} \tag{5.2.14}$$

$$\begin{aligned} V = (19,11,3)(20,12,1)(21,10,2)(22,13,4)(23,14,5) \\ (24,15,6)(25,18,8)(26,16,9)(27,17,7), \end{aligned} \tag{5.2.15}$$

which again are labeled in keeping with the notation of [Con77] and [Wil84]. Note that N_0 and N_{356} are two order four elements that fix the center block $\{13,14,15\}$, and generate a Q_8 subgroup of P. Adjoining F_{03} we obtain a $Q_8.2$ subgroup preserving but not fixing the center block.

To extend the group $A{:}P$ to a group of the shape $2^7.G_2(2)$ we include any symmetry obtained via the following general method. First pick a ringed quartet containing ∞, and call it the chosen quartet. For example, we may take the quartet containing ∞ and the points of the center block. A given quartet is contained in exactly three dozens, and we choose one of those that contains the chosen quartet. For example, we may take the dozen containing the chosen quartet and having M's in all blocks other than that contained in the chosen quartet. There is a unique partition of any dozen into three disjoint ringed quartets, and there is a unique partition of the points of a dozen into six couples in such a way that the union of any two couples is a quartet. In particular the coupling on the chosen dozen refines each of the three ringed quartets it contains. We now choose one of these ringed quartets that is not the already chosen one, and call it the fixed quartet. We call the remaining quartet the stabilized quartet. We begin to define an element $m' \in SU(\mathfrak{r})$ by decreeing that m' fix the points e_i for i in the fixed quartet. We decree also that m' transpose e_i with $e_{i'}$ for $\{i,i'\}$ a couple in either of the chosen or stabilized ringed quartets, except for the couple $\{\infty,\infty'\}$ containing ∞, for which we insist that m' transpose e_∞ with $-e_{\infty'}$. Now consider the dozens that contain the fixed quartet. There are just three: the chosen dozen and two others, and we call these two the fixed dozens. The points of the fixed dozens complementary to those of the fixed quartet exhaust the 16 points of Δ outside of the chosen dozen, and the couplings on

each of the fixed dozens yield couplings on these 16 points. These couplings have the property that exactly one point of each couple is contained in a block containing a point of the fixed quartet, and the other point lies in a block containing a point of the stabilized quartet. We ask now that m' transpose e_i with $f_{i'}$ whenever $\{i,i'\}$ is a couple outside of the chosen dozen and i is in the fixed quartet. Then the action of m' on all of $\mathfrak{r}$ is determined once we decree that m' commute with multiplication by $\mathbf{i}$. Taking

$$\begin{array}{c} \begin{array}{|c|c|c|}\hline \cdot & \cdot & \cdot \\\hline \cdot & LMR & \cdot \\\hline \cdot & \cdot & \cdot \\\hline \end{array} \\ * \end{array} \quad \begin{array}{c} \begin{array}{|c|c|c|}\hline M & M & M \\\hline M & LMR & M \\\hline M & M & M \\\hline \end{array} \\ * \end{array} \quad \begin{array}{c} \begin{array}{|c|c|c|}\hline M & \cdot & M \\\hline \cdot & \cdot & \cdot \\\hline M & \cdot & M \\\hline \end{array} \\ \cdot \end{array} \tag{5.2.16}$$

to be the chosen quartet, the chosen dozen, and the fixed quartet, respectively, we obtain $m' = m$ where m is described in the following way as a coordinate permutation followed by scalar multiplications on coordinates.

$$\begin{array}{c} \begin{array}{|c|c|c|}\hline \begin{matrix}1\\ \mathbf{i}\quad\mathbf{i}\end{matrix} & \begin{matrix}1\\ -\mathbf{i}\quad -\mathbf{i}\end{matrix} & \begin{matrix}1\\ \mathbf{i}\quad\mathbf{i}\end{matrix} \\\hline \begin{matrix}1\\ -\mathbf{i}\quad -\mathbf{i}\end{matrix} & \begin{matrix}-1\\ 1\quad 1\end{matrix} & \begin{matrix}1\\ -\mathbf{i}\quad -\mathbf{i}\end{matrix} \\\hline \begin{matrix}1\\ \mathbf{i}\quad\mathbf{i}\end{matrix} & \begin{matrix}1\\ -\mathbf{i}\quad -\mathbf{i}\end{matrix} & \begin{matrix}1\\ \mathbf{i}\quad\mathbf{i}\end{matrix} \\\hline \end{array} \\ -1 \end{array} \circ \begin{array}{c} (2,5)(3,12)(4,22)(6,9) \\ (8,17)(10,16)(11,20)(13,\infty) \\ (14,15)(18,27)(21,24)(23,26) \end{array} \tag{5.2.17}$$

The symmetry m is denoted M_3E_{01} in [Wil84].

We set M to be the subgroup of $SU(\mathfrak{r})$ generated by A, P and m, and M is then a non-split extension of the form $2^7.G_2(2)$. When written with respect to the basis $\{a_i\}$, the matrices representing the action of M on $\mathfrak{r}$ all have the property that any row or column has exactly one entry, and that entry lies in $\{\pm 1, \pm\mathbf{i}\}$. We call such a matrix monomial, and we call M the monomial group. By replacing the non-zero entries of the monomial matrices representing M with 1's, each monomial matrix becomes a permutation matrix, and we obtain a homomorphism of groups $M \to \overline{M}$ where $\overline{M}$ acts as permutations on the 28 coordinates a_i. The kernel of this homomorphism is the 2^7 subgroup of M, and $\overline{M}$ is isomorphic to $G_2(2)$. We write $\bar{m}$ for the image of $m \in M$ under the map $M \to \overline{M}$.

5.2.3 Monomial invariants

In this section we consider the invariants of the monomial group M in the space $\mathrm{CM}(\mathfrak{s})^0_X$ (see §3.3). In §5.3 we will find that this space is just the subspace of A_{Ru} with degree 7/2.

Recall that the set Δ has been enumerated in §5.2.2, and let us agree to write e_I for $e_{i_1}\cdots e_{i_k}$ whenever $I = \{i_1,\dots,i_k\} \subset \Delta$ and $i_1 < \cdots < i_k$. (We decree that $i < \infty$ for all $i \in \Delta \setminus \infty$.) An orthonormal basis for $\mathrm{CM}(\mathfrak{s})_X$ is then given by the set $\{e_I 1_X\}$ where I ranges over all subsets of Δ.

Suppose that t is an M invariant vector in $\mathrm{CM}(\mathfrak{s})_X$ and that t has non-zero projection onto e_I for some $I \subset \Delta$. Then I must lie in $\mathcal{D}^\circ$, the dual to the dozens code $\mathcal{D}$, since otherwise some element of A would negate e_I. Indeed, t must have non-zero projection also onto $e_{\bar{m}(I)}$ for every $\bar{m} \in \overline{M}$ since $m(e_I 1_X) \in \mathbb{C}e_{\bar{m}(I)}1_X$ for any $m \in M$ with image $\bar{m}$ in $\overline{M}$. (In fact one can show that $m(e_I 1_X) \in \{\pm e_{\bar{m}(I)}1_X\}$ for all $m \in M$ and $I \in \mathcal{D}^\circ$.) In this way we see that M-invariant vectors may be obtained by specifying an orbit $\mathcal{O}$ of $\overline{M} \cong G_2(2)$ on $\mathcal{D}^\circ$ and a map $\gamma : \mathcal{O} \to \mathbb{C}$ such that $m(\gamma_I e_I 1_X) = \gamma_{\bar{m}(I)} e_{\bar{m}(I)} 1_X$ for $m \in M$ and $I \in \mathcal{O}$, and any M-invariant vector in $\mathrm{CM}(\mathfrak{s})_X$ must be a linear combination of vectors obtained in this way from orbits of $\overline{M}$ in $\mathcal{D}^\circ$.

It turns out that a suitable map γ exists only for certain orbits of $\overline{M}$ in $\mathcal{D}^\circ$. For example, explicit calculation reveals that there are 80 orbits of $\overline{M}$ on words of weight 14 in $\mathcal{D}^\circ$, and just 68 of these orbits give rise to M-invariant vectors in the weight 14 subspace of $\mathrm{CM}(\mathfrak{s})_X$. We collect these 68 orbits on weight 14 words in $\mathcal{D}^\circ$ for which suitable maps γ exist into a set $\mathfrak{O} = \{\mathcal{O}\}$, so that the vectors $t_{\mathcal{O}} = \sum_{I \in \mathcal{O}} \gamma_I e_I 1_X$ for $\mathcal{O}$ in $\mathfrak{O}$ span the point-wise M-invariant subspace of $\bigwedge^{14}(\mathfrak{r}) \subset \mathrm{CM}(\mathfrak{s})_X$. We refer to the orbits of $\overline{M}$ in $\mathfrak{O}$ as the *relevant orbits*.

The functions γ and the vectors $t_{\mathcal{O}}$ are determined only up to scalar factors, and for a given I in a relevant orbit $\mathcal{O}$, we may assume if we wish that γ is normalized so that $\gamma_I = 1$. Indeed, the $\overline{M}$ invariant vector $t_{\mathcal{O}}$ is completely determined once we specify an I in $\mathcal{O}$, and insist that $\gamma_I = 1$.

If $\mathcal{O}$ is an orbit of $\overline{M}$ on weight 14 words, then $\mathcal{O}' = \{I' = \Delta \setminus I \mid I \in \mathcal{O}\}$ is also an orbit for $\overline{M}$ on weight 14 words, and the 68 relevant orbits $\mathfrak{O}$ constitute 34 pairs of the form $\{\mathcal{O}, \mathcal{O}'\}$. In the first column of Table 6 we provide a list of 34 weight 14 subsets of Δ, chosen so that the corresponding orbits under $\overline{M}$ constitute one orbit each from the 34 pairs $\{\mathcal{O}, \mathcal{O}'\}$ in $\mathfrak{O}$. The corresponding $\overline{M}$ invariants $t_{\mathcal{O}}$ and $t_{\mathcal{O}'}$ are determined once we insist that $\gamma_I = \gamma_{I'} = 1$ for each I in Table 6, where $I' = \Delta \setminus I$.

From the 68 dimensional space spanned by the $t_{\mathcal{O}}$ and $t_{\mathcal{O}'}$ we require to pick a single vector that will allow us to realize the Rudvalis group. We chose ϱ by setting

$$\varrho = \frac{1}{\sqrt{C}} \sum r_{\mathcal{O}} t_{\mathcal{O}} + r_{\mathcal{O}'} t_{\mathcal{O}'} \tag{5.2.18}$$

where the vectors $t_{\mathcal{O}}$ and $t_{\mathcal{O}'}$ are determined by the entries in the first column of Table 6 as explained above, the coefficients $r_{\mathcal{O}}$ and $r_{\mathcal{O}'}$ are given in the second column of Table 6, and the constant C is given by $C = 86272 = 2^8.337$.

Let F be the subgroup of $SL(\mathfrak{r})$ that fixes ϱ, and recall that $\mathfrak{z}$ denotes the element $e_\Delta e_{\Delta'} \in SL(\mathfrak{r})$, with the ordering chosen so that $\mathfrak{z}$ lies in F. Then F contains the monomial group M. We have the following

Theorem 5.8. *The group F contains a group isomorphic to $2.Ru$.*

Recall that the monomial group M is a maximal subgroup of $2.Ru$ so we require just one more generator fixing ϱ in order to extend M to the desired group. Following [Wil84] we present here an element z (denoted there by E_{24}) that satisfies our objective, and whose action is determined as follows. The element z acts with order two, fixing a 24 dimensional space in $\mathfrak{r}$ and negating the complementary space. The fixed 24 dimensional space is spanned by the vectors of Table 7 along with their images under multiplication by $\mathbf{i}$ and the $Q_8.2$ group of coordinate permutations

Table 6: Data for ϱ

$\overline{M}$ orbit representatives	$r_{\mathcal{O}}$	$r_{\mathcal{O}'}$
$\{3,5,6,8,11,18,19,20,21,22,25,26,27,\infty\}$	$4-3\mathbf{i}$	$4+3\mathbf{i}$
$\{2,3,4,6,7,8,10,12,13,14,17,18,21,\infty\}$	$-4\mathbf{i}$	$4\mathbf{i}$
$\{2,4,5,6,7,8,9,13,14,15,16,22,23,\infty\}$	-2	2
$\{1,5,6,10,12,13,14,15,16,17,20,22,25,27\}$	-2	-2
$\{1,3,9,10,11,12,18,19,20,21,24,26,27,\infty\}$	$-2\mathbf{i}$	$2\mathbf{i}$
$\{1,3,7,8,9,10,12,17,19,20,21,25,27,\infty\}$	$2\mathbf{i}$	$-2\mathbf{i}$
$\{2,3,4,5,6,13,14,15,16,17,19,21,22,25\}$	$-2+4\mathbf{i}$	$2+4\mathbf{i}$
$\{1,2,4,5,7,8,9,10,11,12,13,18,21,\infty\}$	2	2
$\{4,6,8,10,13,15,18,20,21,22,24,26,27,\infty\}$	$-\mathbf{i}$	$-\mathbf{i}$
$\{3,4,7,8,9,10,11,12,13,17,19,23,25,26\}$	$-\mathbf{i}$	$\mathbf{i}$
$\{1,2,3,5,6,9,12,13,15,16,20,22,24,\infty\}$	$\mathbf{i}$	$\mathbf{i}$
$\{1,3,5,6,7,9,11,12,13,15,20,22,24,\infty\}$	$2+\mathbf{i}$	$-2+\mathbf{i}$
$\{1,3,5,6,9,12,14,15,17,19,20,22,24,\infty\}$	$-2-\mathbf{i}$	$-2+\mathbf{i}$
$\{1,2,5,6,7,11,13,14,15,16,20,22,24,\infty\}$	$2\mathbf{i}$	$2\mathbf{i}$
$\{1,2,3,5,7,9,11,13,14,15,16,17,19,\infty\}$	$-\mathbf{i}$	$-\mathbf{i}$
$\{1,4,5,8,10,12,13,20,21,22,23,25,26,27\}$	-2	-2
$\{1,3,5,6,9,10,13,17,18,19,21,24,26,27\}$	$2\mathbf{i}$	$-2\mathbf{i}$
$\{4,8,10,12,15,18,20,21,22,23,25,26,27,\infty\}$	$\mathbf{i}$	$-\mathbf{i}$
$\{1,3,4,5,7,8,9,11,13,14,17,19,21,27\}$	2	2
$\{8,9,14,15,16,17,18,19,21,23,24,25,27,\infty\}$	$-3-2\mathbf{i}$	$3-2\mathbf{i}$
$\{8,9,14,15,16,17,18,19,20,22,23,26,27,\infty\}$	$-3-\mathbf{i}$	$3-\mathbf{i}$
$\{8,9,14,15,16,17,18,19,20,21,22,24,27,\infty\}$	$-2-\mathbf{i}$	$2-\mathbf{i}$
$\{8,9,14,16,18,19,20,21,22,23,25,26,27,\infty\}$	-2	2
$\{9,13,14,16,17,18,19,20,21,23,24,25,27,\infty\}$	$-1+2\mathbf{i}$	$1+2\mathbf{i}$
$\{9,13,14,15,16,18,19,21,22,23,25,26,27,\infty\}$	$1+2\mathbf{i}$	$-1+2\mathbf{i}$
$\{9,13,14,15,16,18,19,20,21,22,23,24,26,27\}$	$-1+\mathbf{i}$	$-1-\mathbf{i}$
$\{8,9,13,14,15,18,19,20,23,24,25,26,27,\infty\}$	-1	-1
$\{8,9,13,14,15,18,19,20,21,22,23,24,26,\infty\}$	$-1+\mathbf{i}$	$1+\mathbf{i}$
$\{8,9,13,15,16,18,19,20,21,22,23,25,26,27\}$	$1-\mathbf{i}$	$-1-\mathbf{i}$
$\{9,12,14,15,16,17,18,20,21,23,24,25,26,27\}$	$-1-\mathbf{i}$	$-1+\mathbf{i}$
$\{9,12,14,15,16,17,18,19,20,21,22,23,25,26\}$	$1-\mathbf{i}$	$1+\mathbf{i}$
$\{9,12,15,17,18,19,20,21,22,23,24,26,27,\infty\}$	$1+\mathbf{i}$	$1-\mathbf{i}$
$\{8,9,12,14,16,17,20,21,22,23,24,25,26,27\}$	$1+3\mathbf{i}$	$-1+3\mathbf{i}$
$\{8,9,12,14,15,16,19,20,21,24,25,26,27,\infty\}$	$2-2\mathbf{i}$	$-2-2\mathbf{i}$

Table 7: Vectors fixed by z

$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
$\cdot$ $\cdot\ \cdot$	1 $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
	1	

$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
$\cdot$ $\cdot\ \cdot$	-1 $\mathbf{i}\ \mathbf{i}$	$\cdot$ $\cdot\ \cdot$
$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
	1	

$\cdot$ $\cdot\ \cdot$	1 $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
1 $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$	1 $\cdot\ \cdot$
$\cdot$ $\cdot\ \cdot$	1 $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
	$\cdot$	

$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ 1$
$\cdot$ $\cdot\ -1$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ -1$
$\cdot$ $\cdot\ 1$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \cdot$
	$\cdot$	

$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ \mathbf{i}$	$\cdot$ $\cdot\ \cdot$
$\cdot$ $\cdot\ -1$	$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ 1$
$\cdot$ $\cdot\ \cdot$	$\cdot$ $\cdot\ -\mathbf{i}$	$\cdot$ $\cdot\ \cdot$
	$\cdot$	

stabilizing the center block. (Recall from §5.2.2 that this $Q_8.2$ group is generated by N_0, N_{356} and F_{03}.) It is a computation to verify that the action of z fixes ϱ. Once again, the computer algebra package [GAP05] is a very useful tool for carrying out the necessary calculations.

Remark. For the interested reader we remark that other elements like z may be defined in the following way. Similar to the case with m in §5.2.2 we choose a ringed quartet containing ∞ and a dozen containing that quartet. We then take N and N' to be two coordinate permutations of order 4 generating the Q_8 group in H that fixes the chosen quartet. We may assume that N stabilizes the ringed quartets contained in the chosen dozen, and that N' interchanges the two quartets other than the chosen quartet. We take F to be an order two permutation extending the Q_8 to $Q_8.2$, and stabilizing the chosen quartet. As above we take Q to be the order three coordinate permutation in H that stabilizes all the ringed quartets containing ∞. We now define an element $z' \in SO(\mathfrak{r})$ by first decreeing that z' fix the elements $e_\infty + e_{\infty'}$ and $e_\infty - e_{\infty'} + f_x + f_{x'}$ if $\{\infty, \infty'\}$ and $\{x, x'\}$ are the couples of the chosen dozen contained in the chosen quartet. We then decree that z' fix $e_i + e_{i'} + e_{i''} + e_{i'''}$ when $\{i, i', i'', i'''\}$ is one of the ringed quartets in the chosen dozen other than the chosen quartet. We insist also that z' fix $Q(e_i + e_{i'}) - N'Q(e_i + e_{i'})$ and $Q(e_i - e_{i'}) + NQ(f_i - f_{i'})$ when $\{i, i'\}$ is a couple of the chosen dozen not in the chosen quartet. Finally, the action of z' on $\mathfrak{r}$ is determined completely once we decree that z' commute with I, N, N' and F. With the chosen quartet and the chosen dozen given by

$$\begin{array}{c}\begin{array}{|c|c|c|}\hline \cdot & \cdot & \cdot \\ \hline \cdot & LMR & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array}\\ * \end{array} \quad \text{and} \quad \begin{array}{c}\begin{array}{|c|c|c|}\hline M & M & M \\ \hline M & LMR & M \\ \hline M & M & M \\ \hline \end{array}\\ * \end{array} \tag{5.2.19}$$

respectively, we may take $N = N_0$, $N' = N_{356}$, and $F = F_{03}$, and we then obtain $z' = z$.

5.3 Construction

Let $\mathfrak{r}$ be as in §5.1 or §5.2 so that $\mathfrak{r}$ is a complex vector space of dimension 28 with positive definite Hermitian form $(\,\cdot\,,\cdot)$ and an orthonormal basis $\{a_i\}_{i\in\Delta}$ indexed by Δ. We write $\mathfrak{r}^*$ for the dual space to $\mathfrak{r}$ with dual basis $\{a_i^*\}_{i\in\Delta}$, and we set $\mathfrak{s} = \mathfrak{r}\oplus\mathfrak{r}^*$. We extend the Hermitian form on $\mathfrak{r}$ in the natural way to $\mathfrak{s}$, and similarly to §3.5, we equip $\mathfrak{s}$ with the symmetric bilinear form denoted $\langle\cdot,\cdot\rangle$ which is just the natural form arising from the pairing between $\mathfrak{r}$ and $\mathfrak{r}^*$ scaled by a factor of 1/2. We also suppose that $\mathcal{E} = \{e_i, e_{i'}\}_{i\in\Delta}$ is the orthonormal basis, with respect to the bilinear form $\langle\cdot,\cdot\rangle$, on $\mathfrak{s}$ determined as in §3.5, and that X is the subgroup of $Spin(\mathfrak{s})$ generated by the expressions $\mathbf{i}e_ie_{i'}$.

The space underlying A_{Ru} is defined by setting $A_{Ru} = A(\mathfrak{s})^0 \oplus A(\mathfrak{s})^0_{\theta,X}$. Then A_{Ru} admits a natural structure of VOA, as demonstrated in §4.2. Let us set $\Omega_{Ru} = \{\omega,\nu_+,\nu_-,\varrho\}$, where ϱ is as in §5.1 or §5.2. (By character table computations we know that there is a unique invariant for the Rudvalis group in $\bigwedge^{14}(\mathfrak{r})$, so the two constructions can differ only up to a constant, and this does not effect the enhanced conformal structure.) Recalling Proposition 4.11 and the remarks following, we then have

Theorem 5.9. *The quadruple $(A_{Ru}, Y, \mathbf{1}, \Omega_{Ru})$ is a self-dual enhanced $U(1)$–VOA of rank 28.*

5.4 Symmetries

We now consider the automorphism group of the enhanced VOA A_{Ru} with enhanced conformal structure determined by Ω_{Ru}.

Let F denote the subgroup of $SL(\mathfrak{r})$ that fixes ϱ. By Theorem 5.6 or Theorem 5.8, a group isomorphic to $2.Ru$ is contained in F. We now show

Proposition 5.10. *The group F is finite.*

Proof. The group F is the stabilizer of a subspace of the natural module for the algebraic group $SL_{28}(\mathbb{C})$, and is thus also algebraic. Since F contains the group R which acts irreducibly on the natural module, we have that F is reductive. Let $\mathfrak{k}$ be the Lie algebra of the connected component of the identity in F. Then $\mathfrak{k}$ embeds in the Lie algebra of $\mathrm{Aut}\,(A_{Ru},\{\omega,\nu_+,\nu_-\})$ which we denote $\mathfrak{g}$, and which may be identified with the subspace of $(A_{Ru})_1$ spanned by the $x_{ij} = a_i(-\frac{1}{2})a_j^*(-\frac{1}{2})\mathbf{1}$ for $i\neq j$, and by the $y_{ij} = x_{ii} - x_{jj}$ for $i \neq j$ (c.f. Proposition 4.10). Then for all $x\in\mathfrak{k}$ we have $\exp(x_{(0)})\varrho = \varrho$, and this implies $x_{(0)}\varrho = 0$ for some non-trivial $x\in\mathfrak{k}$ so long as $\mathfrak{k}$ is not trivial. Consider now the image of $\mathfrak{g}$ in $\bigwedge^{14}(\mathfrak{a})\mathbf{1}_X \subset \tilde{A}(\mathfrak{u})_{7/2}$ under the map $\phi : x \mapsto x_{(0)}\varrho$. The group R acts irreducibly on $\mathfrak{g}$, so ϕ is either the zero map, or is an embedding of R modules. In the former case ϱ is invariant for the action of $SL(\mathfrak{a})$ on $\bigwedge^{14}(\mathfrak{a})\mathbf{1}_X$, but we know that this group acts irreducibly on this space. In the latter case no non-trivial element x of $\mathfrak{g}$ can satisfy $x_{(0)}\varrho = 0$, and thus we have that $\mathfrak{k}$ is trivial. We conclude that $\dim(F) = 0$, and thus F is finite. □

We complete the determination of $\mathrm{Aut}(A_{Ru},\Omega_{Ru})$ by showing that $R \cong 2.Ru$ is a maximal finite subgroup of $SL_{28}(\mathbb{C})$ up to a group of scalar matrices. The techniques we employ in order to establish this result are borrowed directly from [NRS01], including the following

Lemma 5.11. *Suppose X is a finite subgroup of $SL(\mathfrak{r})$ containing R. Then any normal p-subgroup of X is central.*

Proof. Observe that R acts *primitively* on $\mathfrak{r}$, meaning that there is no decomposition $\mathfrak{r} = \mathfrak{r}_1 \oplus \cdots \oplus \mathfrak{r}_k$ into (non-trivial) subspaces that is preserved by R; similarly then for X. If N is a normal subgroup of X, then X permutes the isotypic components of $\mathfrak{r}|_N$ (which is $\mathfrak{r}$ viewed as an N module) and it follows that $\mathfrak{r}|_N$ is isotypic (a direct sum of copies of a single irreducible representation) for N whenever N is normal in X. Since an irreducible representation of an abelian group is faithful only on a cyclic subgroup (and X acts faithfully on $\mathfrak{r}$ by definition), any abelian normal subgroup of X is cyclic.

Let N be a normal p-subgroup of X, and consider the group $C = C_X(N)$, the centralizer of N in X. The intersection $C \cap R$ is normal in R, and contains the center $Z(R)$ of R, and thus $C \cap R$ is either $Z(R)$ or R. In the former case we obtain an embedding $Ru \hookrightarrow \mathrm{Aut}(N)$. In the latter case N can consist only of scalar matrices, since R acts absolutely irreducibly on $\mathfrak{r}$. So we may assume that N is a normal p-subgroup such that $\mathrm{Aut}(N)$ admits an embedding of Ru. An irreducible representation of N in $\mathfrak{r}|_N$ has degree that is a power of p and divides $\dim(\mathfrak{r}) = 28$. So if p is not 2 or 7, then N is abelian, and we have seen above that such a group is cyclic. Evidently, this contradicts $Ru \hookrightarrow \mathrm{Aut}(N)$. For the remaining cases that $p = 2$ (resp. $p = 7$) and N admits a faithful embedding in $SL_4(\mathbb{C})$ (resp. $SL_7(\mathbb{C})$), we consult Hall's classification of p-groups of symplectic type (c.f. [Asc00, (29.3)]). A p-group is said to be of *symplectic type* if it has no non-cyclic characteristic abelian subgroups. N is evidently a group of symplectic type since a characteristic abelian subgroup of N would be normal in X, and we have seen above that such a group is cyclic. Hall's classification reveals that none of the remaining cases for N admit a non-trivial action by Ru as automorphisms. This completes the proof. □

The proof of the following proposition owes a great deal to the proof of Theorem 6.5 in [NRS01].

Proposition 5.12. *If X is a finite subgroup of $SL_{28}(\mathbb{C})$ containing R then $X = \langle R, \xi\,\mathrm{Id}\,\rangle$ for ξ a root of unity in $\mathbb{C}$.*

Proof. Let K be a minimal abelian number field containing $\mathbf{i}$ such that X is conjugate to a subgroup of $SL_{28}(K)$, and let $\mathfrak{O}$ be the ring of integers in K. The group X must preserve a Hermitian lattice in the natural module (which we denote K^{28}). Any Hermitian $\mathbb{Z}[\mathbf{i}]R$ lattice in $\mathbb{Q}(\mathbf{i})^{28}$ is isometric to the Conway–Wales lattice Λ_{Ru} (c.f. [Tie97]); it follows that any $\mathfrak{O}X$ lattice in K^{28} is of the form $I \otimes_{\mathbb{Z}[\mathbf{i}]} \Lambda_{Ru}$ for I a fractional ideal of $\mathfrak{O}$, and if X preserves any such lattice, it must also preserve $\mathfrak{O} \otimes_{\mathbb{Z}[\mathbf{i}]} \Lambda_{Ru}$. Using the action of X on $\mathfrak{O} \otimes_{\mathbb{Z}[\mathbf{i}]} \Lambda_{Ru}$ we may regard X as a group of matrices with entries in $\mathfrak{O}$, and we then may let the Galois group $\Gamma = \mathrm{Gal}(K/\mathbb{Q}(\mathbf{i}))$ act on X, by acting componentwise on the corresponding matrices in $SL_{28}(\mathfrak{O})$.

If Γ is trivial then $K = \mathbb{Q}(\mathbf{i})$ and $X = \langle R, \mathbf{i}\,\mathrm{Id}\,\rangle$, and we are done. Otherwise, let $\mathfrak{p}$ be a prime ideal of $\mathfrak{O}$ that ramifies in $K/\mathbb{Q}(\mathbf{i})$, and let σ be a non-trivial element in the inertia group $\Gamma_{\mathfrak{p}}$.

$$\Gamma_{\mathfrak{p}} = \{\sigma \in \Gamma \mid \sigma(a) \equiv a \pmod{\mathfrak{p}},\ \forall a \in \mathfrak{O}\} \tag{5.4.1}$$

Observe that for arbitrary $g \in X$ we have that $g^{-1}\sigma(g)$ lies in the group $X_{\mathfrak{p}}$ consisting of elements $g \in X$ such that $g \equiv \mathrm{Id} \pmod{\mathfrak{p}}$. The group $X_{\mathfrak{p}}$ is a normal p-subgroup of X, for p the rational

prime divisible by $\mathfrak{p}$. By Lemma 5.11 such a group is central, and we see that the map $g \mapsto g^{-1}\sigma(g)$ is a homomorphism of X into an abelian group. This shows that the commutator subgroup $X^{(1)}$ of X is fixed by σ. Now $X^{(1)}$ is properly contained in X for otherwise K is not minimal, and the argument thus far shows that any finite subgroup of $SL(\mathfrak{a})$ containing R either is realized over $\mathbb{Q}(\mathbf{i})$ (and is then contained in $\langle R, \mathbf{i}\,\mathrm{Id}\,\rangle$), or properly contains its own commutator subgroup. Consider now the chain $X \geq X^{(1)} \geq X^{(2)} \geq \cdots$. Since X is finite and every term $X^{(k)}$ contains $R = R^{(1)}$, the sequence stabilizes and we have $X^{(k)} = X^{(k+1)}$ for some k, and $R \leq X^{(k)} \leq \langle R, \mathbf{i}\,\mathrm{Id}\,\rangle$. This shows that R is normal in X, and we then obtain a map $X \to \mathrm{Aut}(R)$ by letting $x \in X$ act by conjugation on R. The action of x on R is trivial if and only if x commutes with R, and since R acts absolutely irreducibly on K^{28}, such an x must be a scalar matrix in $SL_{28}(K)$. Clearly, the scalar matrices in X constitute the center of X. Let us write T for the group of scalar matrices in X. Then we have an injective map $X/T \to Ru$, which is also surjective since X contains R. We conclude that $X = TR$, as required. □

We are now ready to identify $\mathrm{Aut}(A_{Ru}, \Omega_{Ru})$.

Theorem 5.13. *The group* $\mathrm{Aut}(A_{Ru}, \Omega_{Ru})$ *is a direct product* $7 \times Ru$.

Proof. From Proposition 5.12 we have that F is generated by $R \cong 2.Ru$ together with $\xi\,\mathrm{Id}$ for ξ some root of unity. If $\xi\,\mathrm{Id} \in SL(\mathfrak{r})$ preserves any non-zero element of $\bigwedge^{14}(\mathfrak{a})\mathbf{1}_X \subset (A_{Ru})_{7/2}$ then $\xi^{14} = 1$. Conversely, $\xi\,\mathrm{Id}$ preserves all elements of $\Omega_{Ru} = \{\omega, \nu_+, \nu_-, \varrho\}$ when $\xi^{14} = 1$. So F is a central product $14 \circ R$ with center generated by $\mathfrak{z}$. The image of F in $SL(\mathfrak{r})/\langle\mathfrak{z}\rangle$ is evidently $7 \times Ru$, as required. □

We summarize the results of this section with the following

Theorem 5.14. *The quadruple* $(A_{Ru}, Y, \mathbf{1}, \Omega_{Ru})$ *is a self-dual enhanced* $U(1)$*–VOA of rank* 28. *The full automorphism group of* (A_{Ru}, Ω_{Ru}) *is a direct product of a cyclic group of order seven with the sporadic simple group of Rudvalis.*

Ultimately we would like to characterize A_{Ru} in a fashion analogous to that applied in [Dun07] to the enhanced VOA associated to the Conway group. If U is a self-dual nice rational VOA of rank 28 with no odd vectors of degree less than $7/2$, then the character of U is very restricted, and the method of Proposition 5.7 in [Dun07] can be used to show that the character must coincide with that of A_{Ru}. Probably, the techniques of §5.1 in [Dun07] can be applied to show that the VOAs underlying U and A_{Ru} are isomorphic. We conjecture that self-duality and the above vanishing condition, together with the local algebra (see §2) determined by Ω_{Ru} are sufficient to determine A_{Ru} uniquely (among nice rational enhanced VOAs).

Conjecture. *Suppose that* $(U, Y, \mathbf{1}, \Omega)$ *is a nice rational enhanced VOA with* $\mathcal{A}(\Omega) \cong \mathcal{A}(\Omega_{Ru})$ *such that*

1. *U has rank* 28
2. *U is self-dual*
3. *$U_{\bar{1}}$ has no non-trivial vectors of degree less than* $7/2$

Then $(U, Y, \mathbf{1}, \Omega)$ *is isomorphic to* $(A_{Ru}, Y, \mathbf{1}, \Omega_{Ru})$.

This conjectural characterization of A_{Ru} (and hence the sporadic group Ru) is reminiscent of the uniqueness results that exist for the Golay code, the Leech lattice (see [Con69]), and the enhanced VOA for Conway's group (see [Dun07]), and those which are conjectured to hold for the Moonshine VOA (c.f. [FLM88]) and the Baby Monster VO(S)A (c.f. [Höh96]). All these objects have sporadic automorphism groups. The object A_{Ru} is a first example with non-Monstrous sporadic automorphism group.

6 McKay–Thompson series

In this section we consider the McKay–Thompson series arising from the enhanced VOAs constructed in §§4,5. These series furnish analogues of Monstrous Moonshine for the sporadic group of Rudvalis, and in this section we will derive explicit expressions for them.

The main tool for expressing the series explicitly is the notion of Frame shape, and this is reviewed in §6.1. We then define the McKay–Thompson series associated to the action of a group on a VOA in §6.2, and present explicit expressions for all the McKay–Thompson series arising in our examples.

For $U(1)$–VOAs (see §2) one may consider McKay–Thompson series in two variables, and we define these in §6.4. The enhanced VOA for the Rudvalis group is a $U(1)$–VOA, and so this notion applies also to our main example. As in the ordinary case, one is able to provide explicit expressions for all the two variable McKay–Thompson series arising for the Rudvalis group, and we derive these expressions also in §6.4. For this we employ the notion of weak Frame shape, and this is introduced in §6.3.

Since the Rudvalis group is a non-Monstrous group, the McKay–Thompson series arising are of particular interest. In §6.6 (and the Appendix) we provide some further information about these series.

6.1 Frame shapes

Suppose that $\mathfrak{u}$ is a complex vector space with non-degenerate symmetric bilinear form. Let us set N to be the dimension of $\mathfrak{u}$, and suppose that g is a finite order element of $SO(\mathfrak{u})$ satisfying $g^n = \mathrm{Id}_{\mathfrak{u}}$ for some m. Then g has eigenvalues $\{\xi_i\}_{i=1}^N$ say, where each ξ_i is an m^{th}-root of unity, and the polynomial $\det(\mathrm{Id}_{\mathfrak{u}} - gx)$ satisfies

$$\det(\mathrm{Id}_{\mathfrak{u}} - gx) = \prod_{i=1}^{N} (1 - \xi_i x) \tag{6.1.1}$$

Suppose now that the action of g can be written over $\mathbb{Q}$; that is, suppose that there is some basis with respect to which g is represented by a matrix with entries in $\mathbb{Q}$. (This holds for each element in $2.Ru$, for example, since this group preserves the an integral lattice — the Conway–Wales lattice.) Then all primitive n^{th}-roots of unity in $\{\xi_i\}_{i=1}^N$ appear with the same multiplicity, and there are

uniquely determined integers $m_k \in \mathbb{Z}$ for each k dividing m such that we have

$$\det(\mathrm{Id}_{\mathfrak{u}} - gx) = \prod_{k|n}(1-x^k)^{m_k} \tag{6.1.2}$$

Note also that we have $\sum_{k|n} km_k = N$ in this case. For such g in $SO(\mathfrak{u})$ we may express the data $\{(k, m_k)\}$ by writing

$$g|_{\mathfrak{u}} \sim \prod_{k|n} k^{m_k} \tag{6.1.3}$$

and the formal expression $\prod_{k|n} k^{m_k}$ is called the *Frame shape* for the action of g on $\mathfrak{u}$. We say that g *admits a Frame shape over* $\mathfrak{u}$ if the characteristic polynomial of g^{-1} has an expression of the form (6.1.2).

6.2 Ordinary McKay–Thompson series

Let U be a VOA of rank c and let g be an (VOA) automorphism of U. Then the *McKay–Thompson series associated to the action of g on U* is the q-series defined by

$$\mathrm{tr}|_U\, gq^{L(0)-c/24} = \sum(\mathrm{tr}|_{U_n}\, g)q^{n-c/24} \tag{6.2.1}$$

In the special case that g is the identity we recover, what we call, the *character* of U.

For most of the examples of enhanced VOAs that arise in this article the underlying VOA is of the form $\tilde{A}(\mathfrak{u}) = A(\mathfrak{u})^0 \oplus A(\mathfrak{u})^0_{\theta,E}$ (see §4.2), and the automorphisms all lie in the corresponding group $Spin(\mathfrak{u})/\langle\mathfrak{z}\rangle$. Given g in $Spin(\mathfrak{u})/\langle\mathfrak{z}\rangle$ then, we would like to compute the trace of the operator $gq^{L(0)-c/24}$ on $\tilde{A}(\mathfrak{u})$.

Let $g \in Spin(\mathfrak{u})/\langle\mathfrak{z}\rangle$, and note that g has preimages $\hat{g}$ and $\mathfrak{z}\hat{g}$ say, in $Spin(\mathfrak{u})$, and these in turn have two images in $SO(\mathfrak{u})$; one the negative of the other. We write $g \leftrightarrow \{\pm\bar{g}\}$ for this correspondence between elements of $Spin(\mathfrak{u})/\langle\mathfrak{z}\rangle$ and pairs in $SO(\mathfrak{u})$, and we consider only $g \in Spin(\mathfrak{u})/\langle\mathfrak{z}\rangle$ for which $\pm\bar{g}$ admit Frame shapes over $\mathfrak{u}$. (This will hold for all elements in $2.Ru$, thanks to its action on the Conway–Wales lattice.) Then for $\bar{h}$ either one of $\pm\bar{g}$, suppose that $\bar{h}$ admits a Frame shape $\prod_{k|n} k^{m_k}$ say, where n is the order of $\bar{h}$. Recall $\eta(\tau)$, the Dedekind eta function (0.2.2). Then for such $\bar{h}$ we set

$$\eta_{\bar{h}}(\tau) = \prod_{k|n} \eta(k\tau)^{m_k} \tag{6.2.2}$$

For $\hat{h}$ in $Spin(\mathfrak{u})$ we write $\chi_{\hat{h}}$ for the trace of $\hat{h}$ on the finite dimensional $Spin(\mathfrak{u})$-module $\mathrm{CM}(\mathfrak{u})_E \hookrightarrow A(\mathfrak{u})_\theta$ (see §3.4). We will also convene that $\hat{g}$ be the preimage of g in $Spin(\mathfrak{u})$ whose image in $SO(\mathfrak{u})$ is $\bar{g}$, so that $\mathfrak{z}\hat{g}$ is the preimage of g whose image in $SO(\mathfrak{u})$ is $-\bar{g}$.

We now have the following

Theorem 6.1. *Let $g \in Spin(\mathfrak{u})/\langle\mathfrak{z}\rangle$, with $\hat{g} \in Spin(\mathfrak{u})$ and $\bar{g} \in SO(\mathfrak{u})$ as above. Then the McKay–Thompson series associated to the action of g on $\tilde{A}(\mathfrak{u})$ admits the following expression.*

$$\mathrm{tr}|_{\tilde{A}(\mathfrak{u})} gq^{L(0)-c/24} = \frac{1}{2}\left(\frac{\eta_{\bar{g}}(\tau/2)}{\eta_{\bar{g}}(\tau)} + \frac{\eta_{-\bar{g}}(\tau/2)}{\eta_{-\bar{g}}(\tau)} + \chi_{\mathfrak{z}\hat{g}}\eta_{\bar{g}}(\tau) + \chi_{\hat{g}}\eta_{-\bar{g}}(\tau)\right) \tag{6.2.3}$$

Proof. Let g, $\bar{g}$ and $\hat{g}$ be as in the statement of the theorem. Then $\bar{g}$ has Frame shape $\prod_{k|n} k^{m_k}$ say, and the inverse transformation $\bar{g}^{-1}$ has the same Frame shape (since all primitive k^{th}-roots of unity appear with the same multiplicity for any given k). Let $\{f_i\}_{i=1}^{24}$ be a basis for $\mathfrak{u}$ consisting of eigenvectors of $\bar{g}$ with eigenvalues $\{\xi_i\}_{i=1}^{24}$. Then we have

$$\det(\mathrm{Id}_{\mathfrak{u}} - \bar{g}x) = \prod_i (1-\xi_i x) = \prod_{k|n}(1-x^k)^{m_k} \tag{6.2.4}$$

Recall that $\tilde{A}(\mathfrak{u})$ may be described as $\tilde{A}(\mathfrak{u}) = A(\mathfrak{u})^0 \oplus A(\mathfrak{u})^0_\theta$ (see §3.4, §4.2). We have

$$\mathsf{tr}|_{A(\mathfrak{u})}\mathfrak{z}\hat{g}q^{L(0)-c/24} = q^{-c/24}\prod_{n\geq 0}\prod_i (1-\xi_i q^{n+1/2}) \tag{6.2.5}$$

$$\mathsf{tr}|_{A(\mathfrak{u})_\theta}\mathfrak{z}\hat{g}q^{L(0)-c/24} = q^{c/12}\chi_{\mathfrak{z}\hat{g}}\prod_{n\geq 1}\prod_i (1-\xi_i q^{n}) \tag{6.2.6}$$

(Recall that the image of $\mathfrak{z}\hat{g}$ in $SO(\mathfrak{u})$ is $-\bar{g}$.) Substituting q^r for x in (6.2.4) we obtain

$$\prod_i (1-\xi_i q^r) = \prod_{k|m}(1-(q^k)^r)^{p_k} \tag{6.2.7}$$

so that for $\mathsf{tr}|_{A(\mathfrak{l})}\mathfrak{z}\hat{g}q^{L(0)-c/24}$ for example, we have

$$\begin{aligned}
\mathsf{tr}|_{A(\mathfrak{l})}\mathfrak{z}\hat{g}q^{L(0)-c/24} &= q^{-c/24}\prod_{n\geq 0}\prod_i (1-\xi_i q^{n+1/2}) \\
&= \prod_{k|n}\left(q^{-km_k/48}\prod_{n\geq 0}(1-(q^k)^{n+1/2})^{m_k}\right) \\
&= q^{-c/24}\frac{\eta_{\bar{g}}(\tau/2)}{\eta_{\bar{g}}(\tau)}
\end{aligned} \tag{6.2.8}$$

since the rank c is half the dimension of $\mathfrak{u}$, and thus half the degree of the polynomial $\det(\mathrm{Id}_{\mathfrak{u}} - \bar{g}x)$, so that $2c = \sum_{k|n} km_k$. A similar argument shows that

$$\mathsf{tr}|_{A(\mathfrak{u})_\theta}\mathfrak{z}\hat{g}q^{L(0)-c/24} = q^{c/12}\chi_{\mathfrak{z}\hat{g}}\eta_{\bar{g}}(\tau) \tag{6.2.9}$$

and taking $-\bar{g}$ in place of $\bar{g}$, we find

$$\mathsf{tr}|_{A(\mathfrak{u})}\,\hat{g}q^{L(0)-c/24} = q^{-c/24}\frac{\eta_{-\bar{g}}(\tau/2)}{\eta_{-\bar{g}}(\tau)} \tag{6.2.10}$$

$$\mathsf{tr}|_{A(\mathfrak{u})_\theta}\,\hat{g}q^{L(0)-c/24} = q^{c/12}\chi_{\hat{g}}\eta_{-\bar{g}}(\tau) \tag{6.2.11}$$

Finally, for $g \in Spin(\mathfrak{u})/\langle\mathfrak{z}\rangle$ as above, we have

$$\mathsf{tr}|_{A(\mathfrak{u})^0}\,gq^{L(0)-c/24} = \frac{1}{2}\left(\mathsf{tr}|_{A(\mathfrak{u})}\,\hat{g}q^{L(0)-c/24} + \mathsf{tr}|_{A(\mathfrak{u})}\,\mathfrak{z}\hat{g}q^{L(0)-c/24}\right) \tag{6.2.12}$$

$$\mathsf{tr}|_{A(\mathfrak{u})^0_\theta}\,gq^{L(0)-c/24} = \frac{1}{2}\left(\mathsf{tr}|_{A(\mathfrak{u})_\theta}\,\hat{g}q^{L(0)-c/24} + \mathsf{tr}|_{A(\mathfrak{u})_\theta}\,\mathfrak{z}\hat{g}q^{L(0)-c/24}\right) \tag{6.2.13}$$

since the action of $\mathfrak{z}$ on $A(\mathfrak{u}) \oplus A(\mathfrak{u})_\theta$ coincides with that of θ. This completes the proof. ∎

One may now study the McKay–Thompson series associated to the Rudvalis group as soon as the Frame shapes (of elements in the relevant covering group) are given. These Frame shapes may be deduced easily from the character tables and power maps of these groups, and this data is recorded in [CCN+85]. We list the Frame shapes for elements in the double cover $2.Ru$ in Table 8 under the columns headed SO_{56}. (See also the comments in §6.4.) The character of A_{Ru} will be given in §6.4.

6.3 Weak Frame shapes

Now let us suppose that we are in the situation of §3.5, so that $\mathfrak{u}$ is of the form $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$ for some complex vector space $\mathfrak{a}$ with Hermitian form, and the symmetric bilinear form on $\mathfrak{u}$ is that induced by the natural pairing $\mathfrak{a} \times \mathfrak{a}^* \to \mathbb{C}$, so that any invertible linear transformation of $\mathfrak{a}$ extends naturally to an orthogonal transformation of $\mathfrak{u}$. We consider a finite order element g in $SU(\mathfrak{a})$ with $g^n = \mathrm{Id}_{\mathfrak{a}}$. Then g has eigenvalues $\{\gamma_i\}$ say on $\mathfrak{a}$, where each γ_i is an n^{th}-root of unity and the polynomial $\det(\mathrm{Id}_{\mathfrak{u}} - gx)$ (corresponding to the action of g on $\mathfrak{u}$) satisfies

$$\det(\mathrm{Id}_{\mathfrak{u}} - gx) = \prod_{i=1}^{N} (1 - \gamma_i x)(1 - \gamma_i^{-1} x) \tag{6.3.1}$$

For g in $2.Ru$ it will always be the case that all primitive n^{th}-roots of unity in $\{\gamma_i, \gamma_i^{-1}\}$ appear with the same multiplicity (thanks to the action of this group on an integral lattice — the Conway–Wales lattice) so that we may consider the Frame shape for the action of such an element on $\mathfrak{u}$, as defined in §6.1. Regarding g as a unitary transformation on $\mathfrak{a}$, we are in general not able to find such a nice expression for $\det(\mathrm{Id}_{\mathfrak{a}} - gx)$ — corresponding to the action of g on $\mathfrak{a}$ — as in (6.1.2), but we may consider the following generalization: the notion of weak Frame shape.

For m an integer, k a nonnegative integer, and $a \in \mathbb{Q}/\mathbb{Z}$ we write k_a^m to indicate a factor of the form $(1 - e^{2\pi \mathbf{i} a} x^k)^m$. We write k^m in place of k_0^m so as to incorporate the notation for ordinary Frame shapes, and we agree to multiply the symbols k_a^m just as we do the corresponding polynomials, thus obtaining various identities, including the following.

$$k_a k_a^m = k_a^{m+1} \tag{6.3.2}$$

$$k_{1/p}^m k_{2/p}^m \cdots k_{(p-1)/p}^m = (kp)^m k^{-m} \tag{6.3.3}$$

$$k_{a+1/2}^m = (2k)_{2a}^m k_a^{-m} \tag{6.3.4}$$

We say that $\prod_j (k_j)_{a_j}^{m_j}$ is a *weak Frame shape* for the action of g on $\mathfrak{a}$ when

$$\det(\mathrm{Id}_{\mathfrak{a}} - gx) = \prod_j (1 - e^{2\pi \mathbf{i} a_j} x^{k_j})^{m_j} \tag{6.3.5}$$

Weak Frame shapes always exist, since choosing $a_i \in \mathbb{Q}/\mathbb{Z}$ such that $\gamma_i = e^{2\pi \mathbf{i} a_i}$ for example, we have that $\prod_i 1_{a_i}$ is a weak Frame shape for the action of g on $\mathfrak{a}$.

6.4 Two variable McKay–Thompson series

Let $(U, Y, \mathbf{1}, \{\omega, \jmath\})$ be a $U(1)$–VOA of rank c (see §2), and suppose that the action of $J(0) = \jmath_{(0)}$ on U is diagonalizable. Set $J'(0) = \mathbf{i}J(0)$. We say that $u \in U$ has *charge* m when $J'(0)u = mu$.

Definition. For g an automorphism of $(U, \{\omega, \jmath\})$, the *two variable McKay–Thompson series associated to the action of* g *on* U is the series in variables p and q given by

$$\mathsf{tr}|_U\, g p^{J'(0)} q^{L(0)-c/24} = \sum_{m,n} (\mathsf{tr}|_{U_n^m}\, g) p^m q^{n-c/24} \tag{6.4.1}$$

where U_n^m denotes the subspace of U of degree n consisting of vectors of charge m.

In the limit as $p \to 1$ we recover the (ordinary) McKay–Thompson series associated to the action of g on U. In the case that g is the identity we obtain what we call the *two variable character of* U.

We would like to compute the two variable McKay–Thompson series arising from the action of Ru on A_{Ru}. Observe that this action is contained in that of $SL(\mathfrak{r})/\langle \pm \mathrm{Id} \rangle$, and even in that of $SU(\mathfrak{r})/\langle \pm \mathrm{Id} \rangle$ where $\mathfrak{r}$ is the Hermitian vector space of §5.1 or §5.2, and the VOA underlying A_{Ru} is of the form $\tilde{A}(\mathfrak{s})$ for $\mathfrak{s} = \mathfrak{r} \oplus \mathfrak{r}^*$.

Similar to §6.2, we suppose that g lies in $SU(\mathfrak{a})/\langle \pm \mathrm{Id} \rangle$, and we let $\pm\bar{g}$ be the preimages of g in $SU(\mathfrak{a}) \subset SO(\mathfrak{u})$. For $\bar{h}$ one of $\pm\bar{g}$, one has a weak Frame shape $\prod_j (k_j)_{a_j}^{m_j}$ say, for the action of $\bar{h}$ on $\mathfrak{a}$, curtesy of §6.3. Recall the Jacobi theta function $\vartheta(z|\tau)$ from (0.2.3). For $\bar{h}$ as above, we set

$$\phi_{\bar{h}}(z|\tau) = \prod_j \frac{\vartheta(k_j z + \frac{1}{2} - a_j | k_j \tau)^{m_j}}{\eta(k_j \tau)^{m_j}} \tag{6.4.2}$$

$$\psi_{\bar{h}}(z|\tau) = \prod_j p^{k_j m_j/2} q^{k_j m_j/8} \frac{\vartheta(k_j z + \frac{1}{2}k_j \tau + \frac{1}{2} - a_j | k_j \tau)^{m_j}}{\eta(k_j \tau)^{m_j}} \tag{6.4.3}$$

and we then have

Theorem 6.2. *Let $g \in SU(\mathfrak{a})/\langle \pm \mathrm{Id} \rangle$ with $\pm\bar{g}$ as above. Then the two variable McKay–Thompson series associated to the action of g on $(\tilde{A}(\mathfrak{u}), \{\omega, \jmath\})$ admits the following expression.*

$$\mathsf{tr}|_{\tilde{A}(\mathfrak{u})} g p^{J'(0)} q^{L(0)-c/24} = \frac{1}{2} \left(\phi_{-\bar{g}}(z|\tau) + \phi_{\bar{g}}(z|\tau) + \psi_{-\bar{g}}(z|\tau) + \psi_{\bar{g}}(z|\tau) \right) \tag{6.4.4}$$

Proof. The verification is very similar to that of Theorem 6.1. With g and $\pm\bar{g}$ as above, let $\prod_j (k_j)_{a_j}^{m_j}$ be a weak Frame shape for $\bar{g}$ say, and let $\{\gamma_i\}$ be the eigenvalues for the action of g on $\mathfrak{a}$. Then we have

$$\det(\mathrm{Id}_{\mathfrak{u}} - gx) = \prod_j (1 - e^{2\pi \mathbf{i} a_j} x^{k_j})^{m_j} (1 - e^{-2\pi \mathbf{i} a_j} x^{k_j})^{m_j} \tag{6.4.5}$$

For the action of $-\bar{g}$ on $A(\mathfrak{u})$ and $A(\mathfrak{u})_\theta$ we have

$$\mathsf{tr}|_{A(\mathfrak{u})}(-\bar{g})p^{J'(0)}q^{L(0)-c/24} = q^{-c/24}\prod_{m\geq 0}\prod_i(1-\gamma_i p^{-1}q^{m+1/2}) \times (1-\gamma_i^{-1}pq^{m+1/2}) \tag{6.4.6}$$

$$\mathsf{tr}|_{A(\mathfrak{u})_\theta}(-\bar{g})p^{J'(0)}q^{L(0)-c/24} = p^{c/2}q^{c/12}\prod_{m\geq 0}\prod_i(1-\gamma_i p^{-1}q^{m})(1-\gamma_i^{-1}pq^{m+1}) \tag{6.4.7}$$

since left multiplication by $a_i(-r)$ decreases charge by 1, and left multiplication by $a_i^*(-r)$ increases charge by 1. We substitute $p^{-1}q^r$ or pq^r for x in (6.4.5) for various r, and compare with (6.4.2) and the definition of $\vartheta(z|\tau)$ (0.2.3), so as to obtain

$$\begin{aligned}\mathsf{tr}|_{A(\mathfrak{u})}(-\bar{g})p^{J'(0)}q^{L(0)-c/24} &= q^{-c/24}\prod_{m\geq 0}\prod_j\left(1-e^{2\pi\mathbf{i}a_j}p^{-k_j}q^{k_j(m+1/2)}\right)^{m_j} \\ &\quad\times\left(1-e^{-2\pi\mathbf{i}a_j}p^{k_j}q^{k_j(m+1/2)}\right)^{m_j} \\ &= \phi_{\bar{g}}(z|\tau)\end{aligned} \tag{6.4.8}$$

and similarly,

$$\begin{aligned}\mathsf{tr}|_{A(\mathfrak{u})_\theta}(-\bar{g})p^{J'(0)}q^{L(0)-c/24} &= p^{c/2}q^{c/12}\prod_{m\geq 0}\prod_j\left(1-e^{2\pi\mathbf{i}a_j}p^{-k_j}q^{k_jm}\right)^{m_j} \\ &\quad\times\left(1-e^{-2\pi\mathbf{i}a_j}p^{k_j}q^{k_j(m+1)}\right)^{m_j} \\ &= \psi_{\bar{g}}(z|\tau)\end{aligned} \tag{6.4.9}$$

where we have used the fact that $c=\sum_j k_jm_j$. Finally we note that

$$\begin{aligned}\mathsf{tr}|_{A(\mathfrak{u})^0}\, gp^{J'(0)}q^{L(0)-c/24} &= \frac{1}{2}\Big(\mathsf{tr}|_{A(\mathfrak{u})}\,\bar{g}p^{J'(0)}q^{L(0)-c/24} \\ &\quad+\mathsf{tr}|_{A(\mathfrak{u})}(-\bar{g})p^{J'(0)}q^{L(0)-c/24}\Big)\end{aligned} \tag{6.4.10}$$

$$\begin{aligned}\mathsf{tr}|_{A(\mathfrak{u})^0_\theta}\, gp^{J'(0)}q^{L(0)-c/24} &= \frac{1}{2}\Big(\mathsf{tr}|_{A(\mathfrak{u})_\theta}\,\bar{g}p^{J'(0)}q^{L(0)-c/24} \\ &\quad+\mathsf{tr}|_{A(\mathfrak{u})_\theta}(-\bar{g})p^{J'(0)}q^{L(0)-c/24}\Big)\end{aligned} \tag{6.4.11}$$

and the desired result follows. □

One may now study the two variable McKay–Thompson series associated to the Rudvalis group as soon as the weak Frame shapes (of elements in the double cover) are given. Again, these Frame shapes may be deduced easily from the character tables and power maps of these groups, which are recorded in [CCN+85].

Since they may not be well known, we record the Frame shapes (columns headed SO_{56}) and weak Frame shapes (columns headed SU_{28}) for the Rudvalis group in Table 8. There are 36 conjugacy

Table 8: Frame Shapes for $2.Ru$

Class	SO_{56}	SU_{28}	Class	SO_{56}	SU_{28}
1A	1^{56}	1^{28}	12B	$4^{1}12^{5}/2^{1}6^{1}$	$1^{1}_{1/4}3^{1}_{3/4}12^{2}$
2A	$1^{8}2^{24}$	$1^{4}2^{12}$	13A	$1^{4}13^{4}$	$1^{2}13^{2}$
2B	$4^{28}/2^{28}$	$4^{14}/2^{14}$	14A	$28^{4}/14^{4}$	$28^{2}/14^{2}$
3A	$1^{2}3^{18}$	$1^{1}3^{9}$	14B	$28^{4}/14^{4}$	$28^{2}/14^{2}$
4A	$1^{8}4^{12}$	$1^{4}4^{6}$	14C	$28^{4}/14^{4}$	$28^{2}/14^{2}$
4B	$4^{16}/2^{4}$	$1^{4}_{1/4}4^{6}$	15A	$1^{2}15^{4}/3^{2}$	$1^{1}15^{2}/3^{1}$
4C	$4^{16}/2^{4}$	$4^{8}/2^{2}$	16A	$1^{2}16^{4}/2^{1}8^{1}$	$1^{1}1^{1}_{1/4}2^{1}_{3/4}16^{2}/8^{1}$
4D	$2^{4}4^{12}$	$2^{2}4^{6}$	16B	$1^{2}16^{4}/2^{1}8^{1}$	$1^{1}1^{1}_{3/4}2^{1}_{1/4}16^{2}/8^{1}$
5A	$1^{6}5^{10}$	$1^{3}5^{5}$	20A	$1^{2}4^{2}10^{2}20^{2}/2^{2}5^{2}$	$1^{1}4^{1}10^{1}20^{1}/2^{1}5^{1}$
5B	$5^{12}/1^{4}$	$5^{6}/1^{2}$	20B	$2^{1}4^{1}20^{3}/10^{1}$	$1^{1}_{1/4}2^{1}5^{1}_{3/4}20^{1}$
6A	$1^{2}3^{2}6^{8}$	$1^{1}3^{1}6^{4}$	20C	$2^{1}4^{1}20^{3}/10^{1}$	$1^{1}_{1/4}2^{1}5^{1}_{3/4}20^{1}$
7A	7^{8}	7^{4}	24A	$4^{1}12^{1}24^{2}/2^{1}6^{1}$	$1^{1}_{1/4}3^{1}_{1/4}24^{1}$
8A	$4^{4}8^{6}/2^{4}$	$1^{2}_{1/4}4^{1}8^{3}/2^{1}$	24B	$4^{1}12^{1}24^{2}/2^{1}6^{1}$	$1^{1}_{1/4}3^{1}_{1/4}24^{1}$
8B	$8^{8}/4^{2}$	$1^{2}_{1/4}2^{1}8^{4}/4^{2}$	26A	$4^{2}52^{2}/2^{2}26^{2}$	$4^{1}52^{1}/2^{1}26^{1}$
8C	$4^{2}8^{6}$	$4^{1}8^{3}$	26B	$4^{2}52^{2}/2^{2}26^{2}$	$4^{1}52^{1}/2^{1}26^{1}$
10A	$2^{4}5^{2}10^{4}/1^{2}$	$2^{2}5^{1}10^{2}/1^{1}$	26C	$4^{2}52^{2}/2^{2}26^{2}$	$4^{1}52^{1}/2^{1}26^{1}$
10B	$2^{2}20^{6}/4^{2}10^{6}$	$2^{1}20^{3}/4^{1}10^{3}$	29A	$29^{2}/1^{2}$	$29^{1}/1^{1}$
12A	$1^{2}3^{2}12^{4}$	$1^{1}3^{1}12^{2}$	29B	$29^{2}/1^{2}$	$29^{1}/1^{1}$

classes in the group Ru, and 25 of these conjugacy classes split in two when lifted to $2.Ru$. Suppose that $g \in Ru$ has preimages $\pm\bar{g}$ in $2.Ru < SU(\mathfrak{a})$, and consider the multiplicative map ι on Frame shapes which fixes k_a^m when k is even, and maps k_a^m to $(2k)_{2a}^m k_a^{-m}$ when k is odd. Then for $\prod_j (k_j)_{a_j}^{m_j}$ a weak Frame shape for $\bar{g}$, the image of this under ι is a weak Frame shape for $-\bar{g}$, so we list Frame shapes and weak Frame shapes for only one of $\pm\bar{g} \in 2.Ru$ as $\bar{g}$ ranges over a set of conjugacy class representatives in Ru. Our naming of the conjugacy classes follows [CCN+85].

We record the two variable character of A_{Ru} in the following proposition.

Proposition 6.3. *For the two variable character of A_{Ru} we have*

$$\begin{aligned}\mathsf{tr}|_{A_{Ru}} p^{J'(0)} q^{L(0)-c/24} = {} & \frac{\vartheta(z|\tau)^{28}}{2\eta(\tau)^{28}} + \frac{\vartheta(z+\frac{1}{2}|\tau)^{28}}{2\eta(\tau)^{28}} \\ & + p^{14}q^{7/2}\frac{\vartheta(z+\frac{1}{2}\tau|\tau)^{28}}{2\eta(\tau)^{28}} + p^{14}q^{7/2}\frac{\vartheta(z+\frac{1}{2}\tau+\frac{1}{2}|\tau)^{28}}{2\eta(\tau)^{28}}\end{aligned} \tag{6.4.12}$$

6.5 Two variable modular invariance

Recall that the Jacboi theta function $\vartheta(z|\tau)$ is a holomorphic function on $\mathbb{C}\times\mathbf{h}$, with the Fourier development

$$\vartheta(z|\tau)=\sum_{m\in\mathbb{Z}}p^m q^{m^2/2} \tag{6.5.1}$$

for $p=e^{2\pi\mathbf{i}z}$ and $q=e^{2\pi\mathbf{i}\tau}$ (c.f. (0.2.3)). From (6.5.1) it follows easily that $\vartheta(z+1|\tau)=\vartheta(z|\tau)$, and that $\vartheta(z+\tau|\tau)=p^{-1}q^{-1/2}\vartheta(z|\tau)$.

Definition. For m an integer, we set $\mathfrak{E}_m$ to be the class of functions $f(z|\tau)$ holomorphic on $\mathbb{C}\times\mathbf{h}$ and such that $f(z+1|\tau)=f(z|\tau)$, and such that for given τ we have $f(z+\tau|\tau)=e^{-2\pi\mathbf{i}zm}c(\tau)f(z|\tau)$ for some function $c(\tau)$, holomorphic on $\mathbf{h}$.

Evidently, $\vartheta(z|\tau)$ belongs to $\mathfrak{E}_1$. From Theorem 6.2 and its proof, we see that the two variable character of $(\tilde{A}(\mathfrak{u}),\Omega_U)$ (see §4.3) lies in $\mathfrak{E}_c$ where c is the rank of $\tilde{A}(\mathfrak{u})$, and similarly for $(A(\mathfrak{u}),\Omega_U)$ (see §4.1).

Remark. The class $\mathfrak{E}_m$ is closely related to the class of Jacobi forms of weight 0 and index m (c.f. [EZ85]).

Let us define $\varepsilon(z|\tau)=e^{\pi\mathbf{i}z^2/\tau}$. Then we have

Proposition 6.4. *The following operations generate an action of $SL_2(\mathbb{Z})$ on $\mathfrak{E}_m$.*

$$T: f(z|\tau)\mapsto f(z|\tau+1) \tag{6.5.2}$$
$$S: f(z|\tau)\mapsto \varepsilon(z|\tau)^{-m}f(z/\tau|-1/\tau) \tag{6.5.3}$$

The Poisson summation formula implies the following identity for the Jacobi theta function.

$$\vartheta(z/\tau|-1/\tau)=(-\mathbf{i}\tau)^{1/2}\varepsilon(z|\tau)\vartheta(z|\tau) \tag{6.5.4}$$

On the other hand, one has $\eta(-1/\tau)=(-\mathbf{i}\tau)^{1/2}\eta(\tau)$ for the Dedekind eta function, and from these observations the following proposition quickly follows.

Proposition 6.5. *The two variable character A_{Ru} belongs to $\mathfrak{E}_{28}$, and spans a one dimensional representation of the subgroup of $SL_2(\mathbb{Z})$ generated by S and T^2.*

Given the conclusion of Proposition 6.5, it is natural to ask if Zhu's modular theory for VOAs, and the analogous theory for VOAs due to Dong and Zhao, may be extended to $U(1)$–VOAs, with Jacobi forms on $\mathbb{C}\times\mathbf{h}$ taking up the role played by modular forms in the ordinary case.

6.6 Moonshine beyond the Monster

The terms of lowest charge and degree in the character of A_{Ru} are recorded in Table 9. The column headed m is the coefficient of p^m (as a series in q), and the row headed n is the coefficient of $q^{n-c/24}$ (as a series in p). The coefficients of p^{-m} and p^m coincide, and all subspaces of odd charge vanish.

Table 9: Character of A_{Ru}

	0	2	4	6	8
0	1				
1/2					
1	784	378			
3/2					
2	144452	92512	20475		
5/2					
3	11327232	8128792	2843568	376740	
7/2	40116600	30421755	13123110	3108105	376740
4	490068257	373673216	161446572	35904960	3108105
9/2	2096760960	1649657520	794670240	226546320	35904960
5	13668945136	10818453324	5284484352	1513872360	226546320
11/2	56547022140	45624923820	23757475560	7766243940	1513872360

Many irreducible representations of Ru are visible in the entries of Table 9. For example, we have the following equalities, where the left hand sides are the dimensions of homogeneous subspaces of A_{Ru}, and the right hand sides indicate decompositions into irreducibles for the Rudvalis group.

$$\begin{aligned} 378 &= 378 \\ 784 &= 1 + 783 \\ 20475 &= 20475 \\ 92512 &= (2)378 + 406 + 91350 \\ 144452 &= (3)1 + (3)783 + 65975 + 76125 \\ 376740 &= 27405 + 65975 + 75400 + 102400 \end{aligned} \tag{6.6.1}$$

The identities in (6.6.1) may be regarded as analogues for the Rudvalis group, of the original observations, such as those of (6.6.2)

$$\begin{aligned} 196884 &= 1 + 196883 \\ 21493760 &= 1 + 196883 + 21296876 \\ 864299970 &= (2)1 + (2)196883 + 21296876 + 842609326 \end{aligned} \tag{6.6.2}$$

made by McKay and Thompson, connecting the Monster group with Klein's modular invariant.

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